# 24. On a Generalization of a Theorem of Cox

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1. If $T$ is a real-valued continuous function defined on a compact Hausdorff space $X$ which satisfies

(1) $$\|T^n - I\| \leqq d < 1, \qquad n=0,1,2,\cdots$$

If $|T^n(x)|>1$ for some $x$, then $|T^n(x)| \to +\infty$ as $n\to\infty$ and so contradicts to (1). If $|T^n(x)|<1$ for some $x$, then $|T^n(x)|\to 0$ as $n\to\infty$ and contradicts (1) too. Therefore, (1) implies

(2) $$T=I.$$

Similar proof is also possible for complex-valued case.

If $A$ is a $B^*$-algebra which is commutative and contains the identity, then the Gelfand-Neumark representation theorem gives that $A$ is isometrically isomorphic to the algebra of all continuous complex-valued functions defined on a compact Hausdorff space $X$ which is homeomorphic to the spectrum of $A$, see for instance [2]. Hence, by the above argument, (1) *implies* (2) *for any element* $T$ *of a commutative* $B^*$*-algebra* $A$.

2. An analogous proof is also valid for a commutative semisimple Banach algebra, since the Gelfand representation gives $|T(x)|\leqq\|T\|$ for any maximal ideal $x$. However, the above argument is unable to trace for a commutative Banach algebra having non-zero radical, since $T(x)=1$ for all maximal ideals does not insure (2).

The existence of the non-zero radical is not sufficient to deny the statement that (1) implies (2). For example, if $T$ satisfies (1) and

$$T=I+R, \qquad R^2=0,$$

then the algebra generated by $\{T^n; n=0,1,2,\cdots\}$ contains the non-zero radical which contains at least $R$. In this case (2) is true since otherwise the sphere of radius $d$ centered at the identity contains an unbounded set $\{I+nR; n=0,1,2,\cdots\}$.

3. Very recently, R. H. Cox [1] announces without proof that (1) *implies* (2) *if* $T$ *is a square matrix of finite order being considered as a linear operator defined on a finite dimensional euclidean space.* This is an advance towards the problem under the existence of the radical, since the algebra generated by an arbitrary matrix is not necessarily semisimple.

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