23. *A Note on Congruences*

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The object of this note is to give necessary and sufficient conditions when a collection of disjoint non-empty subsets constitute equivalence classes of a congruence (relation) of a universal algebra. This extends previous results by M. Teissier [4] and G. B. Preston [3].

Let $\boldsymbol{A}=(A, O)$ be a universal algebra with operations $O=\{o_i \mid i \in I\}$. Let Σ be the semigroup with identity of functions generated under composition by all unary functions of the forms $o_i(x, a_1, \cdots, a_{n_i-1})$, $o_i(a_0, \cdots, a_{j-1}, x, a_{j+1}, \cdots, a_{n_i-1})$ $(j=1, \cdots, n_i-2)$, and $o_i(a_0, \cdots, a_{n_i-2}, x)$ for some $i \in I$ and some $a_0, a_1, \cdots, a_{n_i-1} \in A$. The class of an equivalence relation θ containing the element a will be denoted by a/θ.

From [2] recall the following

Proposition 1. *A necessary and sufficient condition for an equivalence relation θ on A in a universal algebra $\boldsymbol{A}$ to be a congruence is that if $(x, y) \in \theta$, then $(\sigma(x), \sigma(y)) \in \theta$ for all $x, y \in A$, and $\sigma \in \Sigma$.*

Definition. A subset $S \subseteqq A$ is *intact* under an equivalence relation θ on A if and only if $S \times S \subseteqq \theta$.

Proposition 2. *A subset $S \subseteqq A$ is intact under an equivalence relation θ on A if and only if $S \subseteqq a/\theta$ for some $a \in A$.*

Proof. For self-containment, we shall give a proof. Let S be intact under θ and $a \in S$ so that $S \times S \subseteqq \theta$. If $x \in S$, then $(x, a) \in \theta$ or $x \in a/\theta$. Thus $S \subseteqq a/\theta$. Conversely, suppose $S \subseteqq a/\theta$ for some $a \in A$. If $(x, y) \in S \times S$, so that $x, y \in S$, then $x, y \in a/\theta$ or (x, a), $(y, a) \in \theta$. By reflexivity of θ then (x, a), $(a, y) \in \theta$, and hence $(x, y) \in \theta$ by transitivity. Therefore $S \times S \subseteqq \theta$.

Theorem 3. *Let $\boldsymbol{A}=(A, O)$ be a universal algebra. The minimum congruence under which each member of a collection $\boldsymbol{S}$ of disjoint non-empty subsets of A is intact is the transitive closure $\theta_S=\bigcup_{i=1}^{\infty} \theta^i$ of the relation $\theta=\{(x, y) \mid x, y \in \sigma(T)$ for some $\sigma \in \Sigma$ and some $T \in \boldsymbol{T}\}$, where $\boldsymbol{T}=\boldsymbol{S} \cup \{\{x\} \mid x \in A \setminus \cup \boldsymbol{S}\}$.*

Proof. Observe that the diagonal of A, $\Delta_A \subseteqq \theta \subseteqq \theta_S$ and $\theta^{-1}=\theta$ so that

$$\theta_S^{-1}=\left(\bigcup_{i=1}^{\infty} \theta^i\right)^{-1}=\bigcup_{i=1}^{\infty} (\theta^i)^{-1}=\bigcup_{i=1}^{\infty} (\theta^{-1})^i=\bigcup_{i=1}^{\infty} \theta^i=\theta_S$$