# 21. Decomposability of Extension and its Application to Finite Semigroups*)

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1. **Introduction.** Let $\mathcal{I}$ be a property preserved by arbitrary homomorphisms, for example, system of identities $\{f_i(x_1, \cdots, x_n) = g_i(x_1, \cdots, x_n); i=1, \cdots, k\}$ where $f_i$ and $g_i$ are words. Let $\rho$ be a congruence on a semigroup $S$. If $S/\rho$ satisfies $\mathcal{I}$ for all $x_1, \cdots, x_n \in S/\rho$, then $\rho$ is called a $\mathcal{I}$-congruence on $S$ and $S/\rho$ is called a $\mathcal{I}$-homomorphic image of $S$. It is well known that given $\mathcal{I}$ and $S$ there is a smallest $\mathcal{I}$-congruence $\rho_0$ on $S$, that is, if $\rho$ is a $\mathcal{I}$-congruence on $S$ then $\rho_0 \subseteq \rho$. $S/\rho_0$ is called the greatest $\mathcal{I}$-homomorphic image of $S$. A semigroup $S$ is called $\mathcal{I}$-indecomposable if the only trivial semigroup is a $\mathcal{I}$-homomorphic image of $S$. In particular if $\mathcal{I} = \{x^2 = x, xy = yx\}$, $\rho$ is called an $s$-congruence, $S/\rho$ is an $s$-homomorphic image of $S$. The study of finite non-simple $s$-indecomposable semigroups is reduced to the study of ideal extensions of an $s$-indecomposable semigroup by an $s$-indecomposable semigroup with zero. From more general point of view we give a few theorems which are applied to the theory of finite $s$-indecomposable non-simple semigroups. The terminology in this paper is based on Clifford and Preston's book.**)

2. **Basic theorems.** First we introduce some notations. Let $\rho$ be a congruence on a semigroup $S$. Let $H$ be a subsemigroup of $S$. $\rho | H$ is the restriction of $\rho$ to $H$.

Let $\xi$ and $\eta$ be congruences on $S$ such that $\xi \subseteq \eta$. We define a congruence $\bar{\eta}$ on $S/\xi$ as follows:

$\bar{x}$ denotes the congruence class (modulo $\xi$) containing $x$

$$\bar{x}\bar{\eta}\bar{y} \text{ if and only if } x\eta y$$

$\bar{\eta}$ is denoted by $\bar{\eta} = \eta/\xi$.

Let $\xi$ be an equivalence on a set $E$ and $A$ be a subset of $E$.

A subset $A \cdot \xi$ of $E$ is defined as follows:

$$A \cdot \xi = \{x \in E; x\xi y \text{ for some } y \in A\}.$$

If $I$ is an ideal of a semigroup $S$ and if $\xi$ is a congruence on $S$, then $I \cdot \xi$ is an ideal of $S$ and $I \subseteq I \cdot \xi$.

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**) A. H. Clifford and G. B. Preston: The algebraic theory of semigroups. Amer. Math. Soc., Providence, R. I. (1961).