17. On a Theorem Concerning Trigonometrical Polynomials

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§1. H. Davenport and H. Halberstan [1] have proved the following theorem from which they have derived a generalization of theorems of K. F. Roth [2] and E. Bombieri [3] on the large sieve:

Thoerem DH1.[1] Let $S_N(x)$ be a trigonometrical polynomial of order N such that

$$S_N(x)=\sum_{n=-N}^{N} c_n e^{inx}$$

and $x_1, x_2, \cdots, x_R$ $(R\geqq 2)$ be distinct points on $(-\pi, \pi)$ such that

$$2\delta=\min_{j\neq k} |x_j-x_k|.$$

Then

$$(1)\qquad \sum_{r=1}^{R} |S_N(x_r)|^2 \leqq 4\cdot 4 \max(N, \pi/2\delta) \sum_{n=-N}^{N} |c_N|^2.$$

Our first theorem is as follows:

Theorem 1. Using the same notation as in Theorem DH1, we have

$$(2)\qquad \sum_{r=1}^{R} |S_N(x_r)|^2 \leqq A \sum_{n=-N}^{N} |c_n|^2$$

for small δ, where $A\leqq 2.34\ (N+\pi/\delta)$ or $A\leqq 3.13\ (N+\pi/2\delta)$.

The inequalities (1) and (2) are mutually exclusive. If N is near to $\pi/2\delta$, then (1) is better than (2), but if they are very different, then (2) is better than (1), except for "small δ."

Further H. Davenport and H. Halberstan [1] proved the following

Theorem DH2. Using the same notation as in Theorem DH1, we have

$$(3)\qquad \sum_{r=1}^{R} |S_N(x_r)|^p \leqq A\sqrt{p}\max(N, 2\pi/\delta)\Big(\sum_{n=-N}^{N} |c_n|^q\Big)^{p/q}$$

where A is an absolute constant and $1/p+1/q=1$, $p\geqq 2$.

Our second theorem is

Theorem 2. Using the same notation as in Theorem DH1,

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1) In [1], Theorem DH1 is stated for the trigonometrical polynomial on the interval $(0,1)$, that is, $S_N=\sum_{n=-N}^{N} c_n e^{2\pi inx}$. Further 2δ in $(-\pi,\pi)$ corresponds to $2\delta/2\pi$ in $(0,1)$.