47. On the Classical Propositional Calculus of A. R. Anderson and N. D. Belnap

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In this paper, we concern with the classical propositional calculus by A. R. Anderson and N. D. Belnap [1]. In their system, axioms are formulated as "if p and $\sim p$ are in a primitive disjunction α, then α is an axiom", and rules of deduction as

$$\text{I)}\ \frac{\varphi(\alpha)}{\varphi(\sim\sim\alpha)},\qquad \text{II)}\ \frac{\varphi(\sim\alpha),\ \varphi(\sim\beta)}{\varphi(\sim(\alpha\vee\beta))}.$$

We shall show that, if we interpret $p\rightarrow q$ as $\sim p\vee q$, then we have Lukasiewicz axiom system:

1) $p\rightarrow(q\rightarrow p)$,
2) $(p\rightarrow(q\rightarrow r))\rightarrow((p\rightarrow q)\rightarrow(p\rightarrow r))$,
3) $(\sim p\rightarrow\sim q)\rightarrow(q\rightarrow p)$.

To prove 1), take

(1) $\sim p\vee(\sim q\vee p)$,

then (1) is an axiom, since (1) contains p and $\sim p$ at the same time. Hence $p\rightarrow(\sim q\vee p)$ and we have $p\rightarrow(q\rightarrow p)$.

To prove 2), it is sufficient to show

(2) $\sim(\sim p\vee\sim q\vee r)\vee\sim(\sim p\vee q)\vee\sim p\vee r$.

The following formulas are axioms:

(3) $\sim r\vee(\sim p\vee r)\vee\sim q$,
(4) $q\vee(\sim p\vee r)\vee\sim q$,
(5) $p\vee(\sim p\vee r)\vee\sim q$.

By the rule of deduction I), (4) implies

(6) $\sim\sim q\vee(\sim p\vee r)\vee\sim q$,

similarly by I), (5) implies

(7) $\sim\sim p\vee(\sim p\vee r)\vee\sim q$.

Then, by II), (3) and (6) imply

(8) $\sim(\sim q\vee r)\vee(\sim p\vee r)\vee\sim q$.

Further, by II), (7) and (8) imply

(9) $\sim(\sim p\vee(\sim q\vee r))\vee(\sim p\vee r)\vee\sim q$.

On the other hand,

(10) $\sim r\vee(\sim p\vee r)\vee p$,
(11) $q\vee(\sim p\vee r)\vee p$,
(12) $p\vee(\sim p\vee r)\vee p$.

are axioms in this system. By I), (10) implies

(13) $\sim r\vee(\sim p\vee r)\vee\sim\sim p$.