# 46. On the Crossed Product of Abelian von Neumann Algebras. II

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1. This note is the continuation of [2].

Previously, we have discussed the equivalence among the groups of automorphisms of an abelian von Neumann algebra due to Dye [3] in connection with the crossed product. In the present note, we shall discuss the another notion introduced by Dye [3], *weak equivalence*, in connection with the crossed product.

We shall use the terminologies and the notations employed in [2].

2. At first, we shall introduce the definition of weak equivalence following after Dye [3].

Let $\mathscr{A}_1$ (resp. $\mathscr{A}_2$) be an abelian von Neumann algebra with the faithful normal trace $\phi_1$ (resp. $\phi_2$) normalized by $\phi_1(1)=1$ (resp. $\phi_2(1)=1$), and $G_1$ (resp. $G_2$) a group of $\phi$-preserving automorphisms of $\mathscr{A}_1$ (resp. $\mathscr{A}_2$). Let $\Psi$ be an isomorphism of $\mathscr{A}_1$ onto $\mathscr{A}_2$ and $\alpha_1$ (resp. $\alpha_2$) be an automorphism of $\mathscr{A}_1$ (resp. $\mathscr{A}_2$). Then, for $A\in\mathscr{A}_2$, $\Psi[(\Psi^{-1}(A))^{\alpha_1}]$ defines an automorphism of $\mathscr{A}_2$ which will be denoted by $\Psi(\alpha_1)$. Similarly, we can define $\Psi^{-1}(\alpha_2)$ on $\mathscr{A}_1$ by $\Psi^{-1}[\Psi(A)^{\alpha_2}]$. Under these circumstances, $G_1$ and $G_2$ are called *weakly equivalent*, if there exists an isomorphism $\Psi$ of $\mathscr{A}_1$ onto $\mathscr{A}_2$ such that $\Psi^{-1}(G_2)=\{\Psi^{-1}(g);\ g\in G_2\}$ is equivalent to $G^1$ in the sense described in [2].

3. In this section, we wish to give a characterization of weak equivalence in the following

**Theorem.** *Let $\mathscr{A}_1$ (resp. $\mathscr{A}_2$) be an abelian von Neumann algebra, $\phi_1$ (resp. $\phi_2$) a normalized faithful normal trace of $\mathscr{A}_1$ (resp. $\mathscr{A}_2$), and $G_1$ (resp. $G_2$) a countable freely acting group of $\phi_1$-(resp. $\phi_2$-) preserving automorphisms of $\mathscr{A}_1$ (resp. $\mathscr{A}_2$). Then a necessary and sufficient condition that $G_1$ and $G_2$ are weakly equivalent is that there exists an isomorphism $\Phi$ of $G_1\otimes\mathscr{A}_1$ onto $G_2\otimes\mathscr{A}_2$ such that*

$$\Phi(\mathscr{A}_1)=\mathscr{A}_2.$$

If $G_1$ and $G_2$ are weakly equivalent, then there exists an isomorphism $\varphi$ of $\mathscr{A}_1$ onto $\mathscr{A}_2$ such that $\varphi^{-1}(G_2)$ is equivalent to $G_1$, by the definition. Hence, by Theorem 1 in [2], there exists an isomorphism $\Phi_1$ of $\varphi^{-1}(G_2)\otimes\mathscr{A}_1$ onto $G_1\otimes\mathscr{A}_1$ such that $\Phi_1(A)=A$ for any $A\in\mathscr{A}_1$.