40. On Pairs of Very-Close Formal Systems

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While we were examining mutual relations between formal systems, we were rather astonished by finding out that there exists a pair of distinct formal systems[1] $\mathbf{M}_1$ and $\mathbf{M}_2$ and another formal system $\mathbf{N}$ stronger than $\mathbf{M}_1$ and $\mathbf{M}_2$ and satisfying the following condition $\mathfrak{C}(\mathbf{M}_1, \mathbf{M}_2, \mathbf{N})$: *For any finite number of propositions* $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$, *the system* $\mathbf{M}_1[\mathfrak{x}_1, \cdots, \mathfrak{x}_n]$ *is equivalent to* $\mathbf{N}$ *if and only if* $\mathbf{M}_2[\mathfrak{x}_1, \cdots, \mathfrak{x}_n]$ *is so, where* $\mathbf{M}_i[\mathfrak{x}_1, \cdots, \mathfrak{x}_n]$ *denotes the formal system stronger than* $\mathbf{M}_i$ *by the axioms* $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$ $(i=1, 2)$.

Any pair of formal systems $\mathbf{M}_1$ and $\mathbf{M}_2$ is called *very-close* if and only if they have such a formal system $\mathbf{N}$ that satisfies $\mathfrak{C}(\mathbf{M}_1, \mathbf{M}_2, \mathbf{N})$. Restricting to formal systems each being stronger than a certain formal system standing on a logic admitting inferences of the implication logic[2] by a finite number of axioms, we can find out a necessary and sufficient condition for any pair of formal systems $\mathbf{M}_1$ and $\mathbf{M}_2$ to be *very-close*. This short note is to exhibit a theorem which gives the condition.

The condition can be stated very simply in the case where the logic has *conjunction* as its logical constant. In this case, any number of axioms can be unified into a single axiom. Here we have: *Two formal systems* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *are very-close if and only if we can find out a formal system* $\mathbf{F}$ *and a pair of propositions* $\mathfrak{p}$ *and* $\mathfrak{q}$ *such that* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *lie between* $\mathbf{F}[\mathfrak{P}]$ *and* $\mathbf{F}[\mathfrak{p}]$, *where* $\mathfrak{P}$ *stands for* $(\mathfrak{p}\rightarrow\mathfrak{q})\rightarrow\mathfrak{p}$.

Taking $\mathfrak{p}$ and $\mathfrak{q}$ as $\mathfrak{p}_1\wedge \cdots \wedge\mathfrak{p}_s$ and $\mathfrak{q}_1\wedge \cdots \wedge\mathfrak{q}_t$, respectively, we can interpret the above theorem even in the case where we do not assume *conjunction* as a logical constant of the logic we stand on. Namely, $\mathbf{F}[\mathfrak{P}]$ and $\mathbf{F}[\mathfrak{p}]$ could be interpreted as $\mathbf{F}[\mathfrak{P}_1, \cdots, \mathfrak{P}_s]$ and $\mathbf{F}[\mathfrak{p}_1, \cdots, \mathfrak{p}_s]$, respectively, for appropriately defind formulas $\mathfrak{P}_1, \cdots, \mathfrak{P}_s$ which would work as $(\mathfrak{p}\rightarrow\mathfrak{q})\rightarrow\mathfrak{p}_i$ $(i=1, \cdots, s)$. This can be interpreted as

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1) Here we call any pair of formal systems *distinct* if and only if there is a proposition which is provable in one of the systems but unprovable in the other.

2) Under the *implication logic*, we understand the logic having *implication* and admitting the following inference rules: (1) $\mathfrak{B}$ is deducible from $\mathfrak{A}$ and $\mathfrak{A}\rightarrow\mathfrak{B}$, (2) $\mathfrak{A}\rightarrow\mathfrak{B}$ is deducible if $\mathfrak{B}$ is deducible from $\mathfrak{A}$. It is the sentential part **LOS** of the primitive logic **LO**. As for **LO**, see [1] Ono.