66. Relations between Complete Integral Seminorms and Complete Volumes

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Let μ be a measure on a σ-ring M. Denote by $v=t\mu$ the function defined by the formula: $v(A)=\mu(A)$ for $A\in V$, where

$$V=\{A\in M:\mu(A)<\infty\}.$$

It is easy to see that the family V is a prering and the function v is a volume. This volume will be called the finite part of the measure μ. If one follows carefully any construction of the space $L_\mu(Y)$ of Lebesgue-Bochner summable functions generated by the measure μ one notices that essentially one needs only the finite part of the measure.

Further observation yields that one needs actually only a functional J which we call a complete integral seminorm. This functional is given by the formula

$$Jf=\int fd\mu\ (f\in L_\mu^+),$$

where L_μ^+ consists of all finite-valued μ-summable nonnegative functions. In this paper we shall find inner characterizations of complete integral seminorms.

If f, g are two real valued functions then by $f\cap g, f\cup g, f\cap 1$ we shall understand the functions $(f\cap g)(x)=inf\{f(x), g(x)\}$, $(f\cup g)(x)=sup\{f(x), g(x)\}$, $(f\cap 1)(x)=inf\{f(x), 1\}$ for all $x\in X$.

We shall write $f\le g$ if $f(x)\le g(x)$ for all $x\in X$. In a similar way we define the relation $f\ge g$.

A sequence f_n is called increasing (decreasing) if the condition $n\le m$ implies $f_n\le f_m$ ($f_n\ge f_m$, respectively).

A nonnegative functional J is called an integral seminorm over the space X if its domain J^+ consists of functions from X into $R^+=\langle 0,\infty)$ and the following three conditions are satisfied:

(1) If $t_1, t_2\in R^+$ and $f_1, f_2\in J^+$ then $t_1f_1+t_2f_2\in J^+$ and

$$J(t_1f_1+t_2f_2)=t_1Jf_1+t_2Jf_2.$$

(2) If $f, g\in J^+$ then $f\cup g\in J^+$ and $f\cap 1\in J^+$.

(3) If $f\le g$ and $f, g\in J^+$ then $g-f\in J^+$.

The integral seminorm is called *upper complete* if, for every increasing sequence $f_n\in J^+$, converging at every point of the space to a finite-valued function f, for which the sequence of numbers Jf_n is bounded, we have $f\in J^+$ and $Jf_n\to Jf$.