63. $B(\mathcal{C})$-Spaces and B-Completeness*)

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Let $\mathcal{C}$ be a class of real or complex locally convex Hausdorff topological vector (abbreviated to l.c.) spaces. An l.c. space E is said to be a $B(\mathcal{C})$-space (resp. $B_r(\mathcal{C})$-space) if, for each F in $\mathcal{C}$, a linear continuous and almost open (resp. one-to-one) mapping of E into F is open (cf: [1], p. 83). If $\mathcal{C}=\mathcal{A}$, the class of *all* l.c. spaces, then $B(\mathcal{A})$-and $B_r(\mathcal{A})$-spaces are respectively called B-complete and B_r-complete spaces (cf: [1], Chap. 7). Similarly, by specializing $\mathcal{C}$, one defines $B(\mathcal{T})$-and $B(\mathcal{Q})$-spaces, where $\mathcal{T}$ and $\mathcal{Q}$ denote the classes of all barrelled and quasibarrelled l.c. spaces, respectively.

It is known ([1], Chapter 7, Theorem 5) that a barrelled $B(\mathcal{T})$-space (resp. $B_r(\mathcal{T})$-space) is B-complete (resp. B_r-complete). In this note, we observe that under suitable conditions an l.c. space E in $\mathcal{C}$ which is also a $B(\mathcal{C})$-space (resp. $B_r(\mathcal{C})$-space) is B-complete (resp. B_r-complete). This extends the preceding result. From this, among other results, we derive a necessary and sufficient condition for a metrizable (resp. normed) l.c. space to be a Fréchet (resp. Banach) space, by using the open mapping theorem. At the end, we note that these ideas can be used more generally for topological spaces. As an instance, we show that the image in a Hausdorff space of a locally compact space under a continuous and almost open mapping is locally compact, thus generalizing a well-known fact.

We shall use the notations and definitions of [1].

Let $\mathcal{C}$ be a class of l.c. spaces. It is understood by the definition of l.c. spaces, that each l.c. space in $\mathcal{C}$ is *Hausdorff*. We say that the class $\mathcal{C}$ satisfies the "invariant" property (I) if the following holds:

If for any l.c. space F there is a linear continuous almost open mapping of a member $E\in\mathcal{C}$ into F, then it follows that F is also in $\mathcal{C}$.

It is known that the classes $\mathcal{T}$ and $\mathcal{Q}$ of barrelled and quasibarrelled l.c. spaces satisfy (I) (cf: [1], p. 20, Prop. 5 and p. 22, (5)(d)). The classes of B-complete l.c. spaces as well as that of S-spaces (cf: [1], Chap. 6) also satisfy (I) (cf: [1], p. 47, and p. 77). We show the following:

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