61. *A Generalization of Durszt's Theorem on Unitary ρ-Dilatations*

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In this paper, an operator means a bounded linear operator on a Hilbert space and we use the notations and terminologies of [1].

Let $C_\rho(\rho\geqq 0)$ denote the class of operators T in a Hilbert space $\mathfrak{H}$, whose powers T^n admit a representation

(1) $$T^n=\rho\cdot PU^n \qquad (n=1,2,\cdots)$$

where U is a unitary operator in some Hilbert space K containing $\mathfrak{H}$ as a subspace and P denotes the projection of $\mathfrak{K}$ onto $\mathfrak{H}$. The following theorems were proved by B.Sz-Nagy and C. Foias in [1].

Theorem A. *An operator T in $\mathfrak{H}$ belongs to the class C_ρ if and only if it satisfies the following conditions:*

(I_ρ) $$\|h\|^2-2\Big(1-\frac{1}{\rho}\Big)\operatorname{Re}(zTh,h)+\Big(1-\frac{2}{\rho}\Big)\|zTh\|^2\geqq 0$$

for $h\in\mathfrak{H}$ and $|z|\leqq 1$.

(II) *The spectrum of T lies in the closed unit disk.*

Theorem B. *C_ρ is a non-decreasing function of ρ in the sense that*

$$C_{\rho_1}\subset C_{\rho_2} \quad \textit{if } 0\leqq\rho_1<\rho_2.$$

These theorems were already proved in [1][2]. Meanwhile E. Durszt [2] has given a simple necessary and sufficient condition for a normal T to belong to C_ρ. In this paper we generalize Durszt's theorem for a suitable class of non-normal operators and show some related results.

Definition 1. An operator T is called a normaloid if $\|T\| = \sup_{\|x\|\leqq 1}|(Tx,x)|$ or equivalently, the spectral radius is equal to $\|T\|$ ([3]—[7]).

Theorem 1. *If T is a normaloid, $T\in C_\rho$ if and only if*

$$\|T\|\leqq\begin{cases}\dfrac{\rho}{2-\rho} & \text{if } 0\leqq\rho\leqq 1\\ 1 & \text{if } \rho\geqq 1.\end{cases}$$

Proof. Let $0\leqq\rho\leqq 1$. In this case (I_ρ) is equivalent with

(I'_ρ) $(2-\rho)\|zTh\|^2-2(1-\rho)\operatorname{Re}(zTh,h)-\rho\|h\|^2\leqq 0$ for $h\in\mathfrak{H}$. $|z|\leqq 1$

That is

(I''_ρ) $(2-\rho)\|Th\|^2\gamma^2-2(1-\rho)|(Th,h)|\gamma\cos\psi-\rho\|h\|^2\leqq 0$

for $h\in\mathfrak{H}$, $0\leqq\gamma\leqq 1$,