58. On the Geometry of G-Structures of Higher Order

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Let $V=R^n$ and V^* its dual. Let M be a differentiable manifold of dimension n and $F^r(M)$ the bundle of r-frames of M. The structure group of $F^r(M)$ is denoted by $G^r(n)$. The Lie algebra $\mathfrak{g}^r(n)$ of $G^r(n)$ is $V\otimes V^*+V\otimes S^2(V^*)+\cdots+V\otimes S^r(V^*)$.

A *transitive graded Lie algebra* is, by definition, a Lie subalgebra $\tilde{\mathfrak{g}}=V+\mathfrak{g}_0+\mathfrak{g}_1+\cdots$ of $V+V\otimes V^*+V\otimes S^2(V^*)+\cdots$, with $\mathfrak{g}_i\subset V\otimes S^{i+1}(V^*)$, satisfying

$$[\mathfrak{g}_i, \mathfrak{g}_j]\subset\mathfrak{g}_{i+j}$$

where $\mathfrak{g}_{-1}=V$.

We call that $\tilde{\mathfrak{g}}$ is of *order* r if

$$\mathfrak{g}_{i+j}\subsetneqq\mathfrak{g}_i^{(j)} \qquad \text{for } i+j<r$$

and

$$\mathfrak{g}_{i+j}=\mathfrak{g}_i^{(j)} \qquad \text{for } i\geqq r \text{ and } j\geqq 0.$$

If $\mathfrak{g}_{k-1}\neq 0$ and $\mathfrak{g}_k=0$ then $\tilde{\mathfrak{g}}$ is said to be of *type* k. In general $r\leqq k+1$.

Let $M_0=\tilde{G}/G$ be a homogeneous space of dimension n. Suppose $\tilde{G}$ is a finite dimensional Lie group whose Lie algebra $\tilde{\mathfrak{g}}$ is a transitive graded Lie algebra of order r and of type k:

$$\tilde{\mathfrak{g}}=V+\mathfrak{g}_0+\cdots+\mathfrak{g}_{s-1}$$

where $s=\operatorname{Max}\{r, k\}$.

We also suppose that G is a closed subgroup of $\tilde{G}$ whose Lie algebra $\mathfrak{g}$ is given by

$$\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1+\cdots+\mathfrak{g}_{s-1}.$$

Then G can be considered as a subgroup of $G^s(n)$.

Definition. Let M be a differentiable manifold of dimension n and G a subgroup of $G^s(n)$ as above. A G-structure $P_G(M)$ of order r and of type k on M is a reduction of $F^s(M)$ to the group G.

Example 1. *Affine structure.* Let $\tilde{G}$ be the affine group and G the isotropy subgroup at the origin so that $\tilde{G}/G$ is the affine space. Then $\tilde{\mathfrak{g}}=V+\mathfrak{gl}(n)=V+V\otimes V^*$ and $\mathfrak{g}=\mathfrak{gl}(n)$. An affine structure on M is, by definition, a reduction of $F^2(M)$ to the group G. Affine structure is a G-structure of order 2 and of trpe 1.

Example 2. *Projective s'ructu' e.* Let $\tilde{G}$ be the group of projective transformations of a real projective space of dimension n and G the isotropy subgroup at the distinguished point so that $\tilde{G}/G$ is