# 82. *A Unique Continuation Theorem for Solutions of the Schrödinger Equations*

By Kyûya MASUDA

Department of Mathematics, University of Tokyo

(Comm. by Zyoiti SUETUNA, M.J.A., May 12, 1967)

**1. Introduction.** We are concerned with the unique continuation property of solutions of the equation

$$i^{-1}\frac{\partial}{\partial t}u=\sum_{j=1}^{n}[i\,\partial/\partial x_j+b_j(x)]^2u+q(x)u \tag{1}$$

for $t$ in $(-\infty, \infty)$, and $x$ in $R^n$, where $b_j(x)$ are real-valued functions of class $C^2$. We assume that the real-valued measurable function $q(x)$ satisfies the conditions (a) $q(x)$ is square-integrable over any compact set in $R^n$, (b) $q\{x\}$ is locally ess. bounded in a connected open set $R^n-Q$, and $Q$ is of measure zero, (c) the operator $H$ is essentially self-adjoint in $\boldsymbol{L}^2=\boldsymbol{L}^2(R^n)$, where $H$ is an operator defined by $\boldsymbol{D}(H)=C_0^\infty(R^n)$, and $H\varphi=\sum_{j=1}^{n}[i\partial/\partial x_j+b_j]^2\varphi+q\varphi$, $\boldsymbol{D}(H)$ being the domain of $H$, (d) the spectral set of the self-adjoint extension $\boldsymbol{H}$ of $H$ is bounded from below.

Some equations, appearing in current structure of quantum mechanics, are of the form (1). However the Schrödinger equations with the Stark effect potential are excluded from our consideration, since the spectral set of their Hamiltonian operators is, generally, not bounded from below (see Kato [1, 2], Stummel [3], Wienholtz [4], Ikebe-Kato [5]). The purpose of the present note is to show the following

**Theorem.** *Let $u$ be a weak solution of* (1). *If $u$ vanishes in some nonempty open subset $G$ of the $(x, t)$-space $R^{n+1}$, then $u$ vanishes identically.*

Here by a *weak solution* of (1) we mean a function $u(\cdot, t)$ of $t$ with the following properties:

(i) $u(\cdot, t)$ takes values in $\boldsymbol{L}^2$, and is strongly continuous for $t$ in $(-\infty, \infty)$ with respect to the norm of $\boldsymbol{L}^2$; (ii) we have

$$\int_{-\infty}^{\infty}\int_{R^n}u(x, t)\,\{i^{-1}\partial/\partial_t\overline{\Phi(x, t)}+\sum_{j=1}^{n}[-i\,\partial/\partial x_j+b_j]^2\overline{\Phi(x, t)}+q(x)\,\overline{\Phi(x, t)}\}dx\,dt=0$$

for any $\Phi$ in $C_0^\infty(R^n\times(-\infty, \infty))$, where $\bar{b}$ is the complex conjugate of $b$.

**2. Proof of the theorem.** Let $u_0(x)=u(x, 0)$. We define

$$S(z)u_0=\int_m^\infty \exp(iz\lambda)dE(\lambda)u_0\ ,$$