103. On Maharam Subfactors of Finite Factors

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1. H. A. Dye [2] has laboriously investigated the structure of measure preserving transformations. In his study, Maharam's lemma plays an eminent role.

It seems natural to consider that a non-commutative version of Maharam's lemma is useful in the theory of von Neumann algebras. We shall introduce a notion of Maharam subalgebra (cf. Definition in § 2), motivated by Maharam's lemma.

In this paper, we shall treat subfactors of II_1-factors which are Maharam subalgebras. Maharam subalgebras in general von Neumann algebras of finite type will be discussed in a subsequent paper.

2. In the first place, we shall state briefly main properties of the conditional expectation of a finite von Neumann algebra introduced and discussed by H. Umegaki [5].

Let $\mathcal{A}$ be a finite factor, then there exists a unique faithful normal trace ϕ on $\mathcal{A}$ such that $\phi(I)=1$. Let $\mathcal{B}$ be a subfactor of $\mathcal{A}$. Then for each A in $\mathcal{A}$, there exists a normal linear mapping $A\to A^{\varepsilon}$ of $\mathcal{A}$ onto $\mathcal{B}$ which has the following properties:

(1) $\phi(AB)=\phi(A^{\varepsilon}B)$, for $A\in\mathcal{A}$ and $B\in\mathcal{B}$,

(2) $A^{\varepsilon}=0$ and $A\geqq 0$ implies $A=0$,

(3) $A\geqq 0$ implies $A^{\varepsilon}\geqq 0$,

(4) $A^{*\varepsilon}=A^{\varepsilon *}$,

(5) $(AB)^{\varepsilon}=A^{\varepsilon}B$, for $A\in\mathcal{A}$ and $B\in\mathcal{B}$,

(6) $I^{\varepsilon}=I$,

(7) $(AB)^{\varepsilon}=(BA)^{\varepsilon}$, for $A\in\mathcal{A}$ and $B\in\mathcal{A}\cap\mathcal{B}'$.

The mapping ε will be called the *conditional expectation* of $\mathcal{A}$ relative to $\mathcal{B}$. The conditional expectation is uniquely determined by (1).

Now, we shall introduce the following

Definition. Let $\mathcal{A}$ be a finite factor, $\mathcal{B}$ a subfactor of $\mathcal{A}$ and ε the conditional expectation of $\mathcal{A}$ relative to $\mathcal{B}$. Then $\mathcal{B}$ is called a *Maharam subalgebra* of $\mathcal{A}$ if for any A in $\mathcal{B}$ such that $0\leqq A\leqq 1$, there exists a projection E in $\mathcal{A}$ such that

$$E^{\varepsilon}=A.$$

The following properties on Maharam subalgebras are clear by