96. Topology of Compact Sasakian Manifolds

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Introduction. The Betti numbers of compact Kählerian manifolds were studied by many geometers. Especially the second Betti numbers of compact Kählerian manifolds were studied by M. Berger [1], R. L. Bishop and S. I. Goldberg [2] and others. It is natural to research these problems in compact Sasakian manifolds, and in fact, they were studied by S.I. Goldberg [5], S. Tachibana [7], S. Tachibana and Y. Ogawa [8] and S. Tanno [10], etc.

1. In [10] we have used $(m-1)$-homothetic deformations to get results of the first Betti numbers. We call these deformations D-homothetic deformations, where D denotes the distribution defined by a contact form η. To get results of the second Betti numbers and harmonic forms, we also utilize a D-homothetic deformation:

$$(1.1) \qquad g \to {}^*g = \alpha g + (\alpha^2 - \alpha)\, \eta \otimes \eta$$

of the associated Riemannian metric g for a positive constant α. Then if (ϕ, ξ, η, g) is a Sasakian structure for a contact form η, then $({}^*\phi = \phi, {}^*\xi = \alpha^{-1}\xi, {}^*\eta = \alpha\eta, {}^*g)$ is also a Sasakian structure. By studying the relations of harmonic forms with respect to g and ${}^*g$, we get

Theorem 1. *A compact m-dimensional Sasakian manifold M with sectional curvature $> -3/(m-2)$ has the first Betti number $b_1(M) = 0$. If $m = 3$, we have also $b_2(M) = 0$.*

A harmonic 2-form w is called of the hybrid type (pure type, resp.) if it satisfies

$$(1.2) \qquad w(\phi X, \phi Y) = w(X, Y) \qquad (= -w(X, Y), \text{resp.})$$

for any vector fields X and Y on M.

Theorem 2. *If $m \geq 5$, a compact Sasakian manifold M with sectional curvature $> -3/(m-2)$ has no harmonic 2-form of the pure type. And if sectional curvature > 0, there is no harmonic 2-form of the hybrid type. Especially, then, we have $b_2(M) = 0$.*

Remark. Under the additional condition that ξ is regular, S. I. Goldberg [5] obtained the last half of Theorem 2. In Kählerian case the similar fact was obtained by R. L. Bishop and S. I. Goldberg [2].

2. We denote by $K(X, Y)$ the sectional curvature for 2-plane determined by X and Y. As is well known in Kählerian manifolds holomorphic pinchings were studied by several authers. In Sasakian