133. A Characterization of Spectraloid Operators and its Generalization

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Normaloid operators are characterized by the equality $\|T^n\| = \|T\|^n$ for every natural number n. We give here a similar characterization of spectraloid operators and coincidently we define two families of new classes of non-normal operators broader than the class of normaloid operators associating with these characterizations. Each family forms an atomic lattice by the set inclusion relation.

In what follows an operator means a bounded linear operator on a complex Hilbert space.

1. For each operator T we associate three non-negative numbers

$$\|T\| = \sup_{\|x\|=\|y\|=1} |(Tx, y)|, \qquad \|T\|_N = \sup_{\|x\|=1} |(Tx, x)|,$$
$$r(T) = \sup\{|\lambda|: \lambda \in \sigma(T)\},$$

(where $\sigma(T)$ is the spectrum of T), which are called the operator norm, numerical radius and the spectral radius of T respectively. These are related by

(1) $$r(T) \leqq \|T\|_N \leqq \|T\|$$

(2) $$r(T) = \lim_{n\to\infty} \|T^n\|_N^{\frac{1}{n}} = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$$

For $\|T\|_N$ the following properties are known

(3) $$\|T\|_N = 0 \quad \text{if and only if } T = 0,$$

(4) $$\|\lambda T\|_N = |\lambda| \, \|T\|_N \quad \text{for every scalar } \lambda,$$

(5) $$\|T+S\|_N \leqq \|T\|_N + \|S\|_N$$

(6) $$1/2\|T\| \leqq \|T\|_N \leqq \|T\| .$$

That is, $\|T\|_N$ is a new norm equivalent to the operator norm $\|T\|$. On the other hand $r(T)$ satisfies (4) but not (3) and (5) remains only in a restricted form. Hence $r(T)$ is not a norm in a strict sense but we may interpret it as a kind of generalized norm.

It is known that these satisfy the same kind of power inequality:

(7) $$\|T^n\| \leqq \|T\|^n, \qquad \|T^n\|_N \leqq \|T\|_N^n, \qquad r(T^n) \leqq r(T)^n,$$

Exactly $r(T^n) = r(T)^n$ for every operator T by the spectral mapping theorem ([1]—[4]).

Following Halmos [2] and Wintner [5], we give

Definition 1. An operator T is called to be spectraloid if

$$\|T\|_N = r(T)$$