131. *The Continuity and the Boundedness of Linear Functionals on Linear Ranked Spaces*

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(Comm. by Kinjiro KUNUGI, M.J.A., Sept. 12, 1967)

1. The definition of a bounded set and its properties. Let E be a linear ranked space, by which name we mean a linear space where $\mathfrak{V}_n$ are defined and satisfy axioms (A), (B), (a), (b), (1), (2), (3),[1)]

Definition 1. A subset B in E is called bounded if, for arbitrary n, there is an m, $m \geq n$, and a $V \in \mathfrak{V}_m$ which absorbs B.

Evidently the subset of a bounded set is also bounded. A set consisting of only one point is bounded (cf. axiom (3)).

The linear sum and the union of bounded sets are bounded, too. In fact, let A and B be bounded. For arbitrary n, we can choose an M such that, if $\lambda \geq M$, $\mu \geq M$, then $\phi(\lambda, \mu) \geq n$. Since A and B are bounded, there are $m_1 \geq M$, $m_2 \geq M$, $V_1 \in \mathfrak{V}_{m_1}$, $V_2 \in \mathfrak{V}_{m_2}$ and $\rho_1 > 0$, $\rho_2 > 0$ with $\rho_1 A \subseteq V_1$, $\rho_1 B \subseteq V_2$. Let $\rho = \min.(\rho_1, \rho_2)$. Then

$$\rho(A+B) \subseteq V_1 + V_2, \qquad \rho(A \cup B) \subseteq V_1 \cup V_2 \subseteq V_1 + V_2.$$

Applying (1) for V_1, V_2, E, there exist an $m \geq \phi(m_1, m_2)$, a $V \in \mathfrak{V}_m$ such that $V_1 + V_2 \subseteq V$. Thus we get an $m \geq n$ and a $V \in \mathfrak{V}_m$ which absorbs $A+B$ and $A \cup B$, and therefore they are bounded.

From the properties just proved, it follows that a finite set is bounded.

Proposition 1. *If* $\{\lim x_n\} \neq \phi$, *then the set* $\{x_n\}$ *is bounded* (i.e. *the convergent sequence makes a bounded set*).

Proof. We may assume $\{\lim x_n\} \ni 0$. In fact, $\{\lim x_n\} \ni x$ is equivalent to $\{\lim(x_n - x)\} \ni 0$. If we show that $\{x_n - x\}$ is bounded, we can assert that $\{x_n\} = \{x_n - x\} + \{x\}$, a linear sum of two bounded sets, is bounded. Let $\{\lim x_n\} \ni 0$. Then there exists a sequence $\{V_n\}$ such that

$$V_n \in \mathfrak{V}_{\alpha_n}, \alpha_n \uparrow \infty, V_n \supseteq V_{n+1}, x_n \in V_n (n=1, 2, \cdots)$$

For arbitrary given N, we can choose an n_0 such that,

$$\phi(m, \alpha_n) \geq N \text{ for } m \geq n_0, n \geq n_0.$$

Let us denote the set $\{x_n\}$ by A, and let $A = A_1 \cup A_2$, where $A = \{x_n; 1 \leq n \leq n_0 - 1\}$, $A_2 = \{x_n; n \geq n_0\}$. Then, $A_2 \subseteq V_{n_0}$. On the other hand, since A_1 is finite and therefore bounded, there is an $m \geq n_0$, a $V \in \mathfrak{V}_m$, and a $\rho > 0$ with $\rho A_1 \subseteq V$.

Let $\rho' = \min.(\rho, 1)$. Then, $\rho A = \rho'(A_1 \cup A_2) \subseteq V \cup V_{n_0} \subseteq V + V_{n_0}$.

1) M. Washihara: On ranked spaces and linearity. Proc. Japan Acad., **43**, 584-589 (1967).