130 On Ranked Spaces and Linearity

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Let E be a linear space over the real or complex numbers, where defined families of subsets $\mathfrak{B}_n (n=0, 1, 2, \cdots,)$ which satisfy following conditions:

(A) For every V in $\mathfrak{B}$, $0 \in V$ (where $\mathfrak{B}=\bigcup_{n=0}^{\infty} \mathfrak{B}_n$).

(B) For U, V in $\mathfrak{B}$ there is a W in $\mathfrak{B}$ such that $W \subseteq U \cap V$.

(a) For any U in $\mathfrak{B}$ and for any integer n, there is an m such that $m \geq n$, and a V in $\mathfrak{B}_m$ such that $V \subseteq U$.

(b) $E \in \mathfrak{B}_0$.

For each point x in E, we shall call $x+V$ a neighbourhood of x with rank n, when $V \in \mathfrak{B}_n$. Then E is a ranked space [1] with indicator ω_0. Furthermore, for any sequence $\{x_n\}$ in E, we have $\{\lim x_n\} \ni x$ [1] if and only if $\{\lim (x_n - x)\} \ni 0$. In fact, if $\{\lim x_n\} \ni x$, there exists a sequence of neighbourhoods of x, $\{v_n(x)\}$, such that

$$v_n(x)=x+V_n,\ V_n \in \mathfrak{B}_{\alpha_n},\ \alpha_n \uparrow \infty,\ v_n(x) \supseteq v_{n+1}(x),\ x_n \in v_n(x).$$

This implies that $V_n \supseteq V_{n+1}$, and therefore $\{\lim (x_n - x)\} \ni 0$. The converse is also obvious.

Now, we set following three axioms concerning the relation between the linear operations and the ranks of neighbourhoods.

(1) There exists a non-negative function $\phi(\lambda, \mu)$, defined for $\lambda \geq 0$ and $\mu \geq 0$, such that $\lim_{\lambda, \mu \to \infty} \phi(\lambda, \mu) = \infty$, and the following holds; if $U \in \mathfrak{B}_l$, $V \in \mathfrak{B}_m$, $W \in \mathfrak{B}_n$, $n \leq \phi(l, m)$, and $U+V \subseteq W$, then, there is an $n^* \geq \phi(l, m)$, and a $W^* \in \mathfrak{B}_{n^*}$ such that $U+V \subseteq W^* \subseteq W$.

(2) There exists a non-negative function $\psi(\lambda, \mu)$, defined for $\lambda \geq 0$ and $\mu \geq 1$ such that $\lim_{\lambda \to \infty} \psi(\lambda, \mu) = \infty$ for each fixed μ, and the following holds; let α be a scalar with $|\alpha| \geq 1$. If $U \in \mathfrak{B}_m$, $V \in \mathfrak{B}_n$, $\alpha U \subseteq V$, and $n \leq \psi(m, |\alpha|)$, then there is an $n^* \geq \psi(m, |\alpha|)$ and a $V^* \in \mathfrak{B}_{n^*}$ such that $\alpha U \subseteq V^* \subseteq V$.

(3) Let $U \in \mathfrak{B}$ and $x \in U$. Then for any n, there is an $m \geq n$, a $V \in \mathfrak{B}_m$ and some positive ρ such that $\rho x \in V \subseteq U$.

Moreover, we assume that every V in $\mathfrak{B}$ is circled (i.e. if $x \in V$ and $|\alpha| \leq 1$, then $\alpha x \in V$).

When E satisfies all these axioms, we can assert that

I. if $\{\lim x_n\} \ni x$ and $\{\lim y_n\} \ni y$, then $\{\lim (x_n + y_n)\} \ni x+y$.

II. if $\{\lim x_n\} \ni x$, then for any scalar λ, $\{\lim \lambda x_n\} \ni \lambda x$.