127. *Notes on Hilbert Transforms of Vector Valued Functions in the Complex Plane and Their Boundary Values*

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1. Introduction. Let $\mathfrak{H}$ be a complex Hilbert space with norm $\| \ \|$ and ρ be a non-negative measure on $R=(-\infty,\infty)$ which is finite on every bounded Borel set. We denote by $L_\rho(R;\mathfrak{H})=L_\rho$ the set of all $\mathfrak{H}$-valued strongly ρ-measurable functions $f(t)$ on R satisfying the condition $\int_{-\infty}^{\infty}\|f(t)\|\,d\rho(t)<\infty$. For each $f(t)\in L_\rho$ we define its real and complex Hilbert transforms $H_\rho[f]$ and $\mathcal{H}_\rho[f]$ with respect to ρ as follows (if the right members exist):

$$H_\rho[f](x)=\text{p.v.}\int_{-\infty}^{\infty}\frac{1}{t-x}f(t)d\rho(t)$$
$$=\text{s}-\lim_{\varepsilon\downarrow 0}\int_{|t-x|>\varepsilon}\frac{1}{t-x}f(t)d\rho(t) \qquad (x\in R), \tag{1}$$

$$\mathcal{H}_\rho[f](z)=\int_{-\infty}^{\infty}\frac{1}{t-z}f(t)d\rho(t) \qquad (\text{im}\, z\neq 0), \tag{2}$$

where the integrals are taken in the sense of Bochner and s$-$lim means the limit in the strong topology of $\mathfrak{H}$. Clearly $\mathcal{H}_\rho[f](z)$ exists for all $z(\text{im}\, z\neq 0)$ and it is a $\mathfrak{H}$-valued analytic function in the upper and lower half-planes. We are concerned with the behavior of $\mathcal{H}_\rho[f](z)$ as z approaches to R from upper or lower half-plane. At first we note some measure theoretic points. Since $f(t)\in L_\rho$ is a strong limit of a sequence of step functions at $\rho-$a.e. (almost every) $t\in R$, there exists a (closed) separable subspace $\mathfrak{H}_f$ of $\mathfrak{H}$ such that $f(t)\in\mathfrak{H}_f$ for $\rho-$a.e. $t\in R$. Therefore we may assume without loss of generality that $\mathfrak{H}$ itself is separable. We put $\rho=\rho_0+\rho_1$ where ρ_0 is the singular part of ρ and ρ_1 is the absolutely continuous part of ρ (with density $\rho'(t)$). Then $f(t)\rho'(t)$ is strongly Lebesgue measurable, and by the standard argument we can see that for a.e. $t\in R$ $\int_{t-h}^{t+h}\|f(s)\rho'(s)-f(t)\rho'(t)\|\,ds=o(h)$ and $\int_{t-h}^{t+h}\|f(s)\|\,d\rho_0(s)=o(h)$ as $h\downarrow 0$. The set of all such t is denoted by $\Lambda_{\rho,f}$. Clearly $(\Lambda_{\rho,f})^c$ is a null set ($=$a set of Lebesgue measure 0) and $\text{s}-\lim_{h\to 0}\frac{1}{h}\int_t^{t+h}f(t)d\rho(t)$ $=f(t)\rho'(t)$ for each $t\in\Lambda_{\rho,f}$.

Now our result reads as follows: