163. *On Extension of Almost Periodic Functions*

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In this note, we shall prove an extension theorem of almost periodic functions on a topological semifield. For the concept of topological semifield, see [1] and [2].

Let E_1 be an arbitrary topological semifield, E_2 a complete topological semifield. We consider the set M of all bounded function $f: E_1 \to E_2$. For $f, g \in M$, we define its distance by

$$\rho(f, g) = \operatorname*{sub}_{x \in E_1} d(f(x), g(x)) = \sup_{x \in E_1} |f(x) - g(x)|$$

where $|x|$ denotes the absolute value of x. As easily seen, $\rho(f, g)$ satisfies the well known axioms on a metric. Then M is a metric space over a topological semifield E_2. E_2 is complete, so M is complete.

Definition 1. A function $f(x)(x \in E_1)$ is called *almost periodic*, if it is continuous on E_1, and if for every neighborhood $U^q_{0,\varepsilon}$ (in E_2) there exists a neighborhood $U^q_{a,a+\delta}$ (in E_1) containing at least one element $y = y(\varepsilon)$ for which the relation $d(f(x+y), f(x)) \in U^q_{0,\dot{\varepsilon}}$ for all $x \leftarrow U^q_{a,a+\delta}$ holds.[1] Such an element $y(\varepsilon)$ is called an ε-period of the function f.

Then every almost periodic function is bounded on the topological semifield and therefore belongs to the space M.

Definition 2. *A* set K of a metric space X over a topological semifield is called *ε-net* for the set M of the space, if for every element $f \in M$ there exists an element $f_\varepsilon \in K$ such that $\rho(f, f_\varepsilon) \in U^q_{0\,\varepsilon}$.

Proposition (Extension of Hausdorff's theorem). In order that a set M in a metric space X over a topological semifield be compact, it is necessary that for every $\varepsilon > 0$ there exists a finite ε-net for M. If the space X is complete, then the condition is also sufficient.

Proof of necessity: We assume that M is compact. Let f_1 be an arbitrary element of M. If $\rho(f, f_1) \in U^q_{0,\varepsilon}$ for all $f \in M$, then a finite ε-net exists. If, however, this is not the case, then there exists an element $f_2 \in M$ such that $\rho(f_1, f_2) \notin U^q_{0,\dot{\varepsilon}}$. If for every element $f \in M$ either $\rho(f_1, f) \in U^q_{0,\varepsilon}$ or $\rho(f_2, f) \in U^q_{0,\varepsilon}$, then we have found a finite ε-net. If, however, this does not hold, then there exists an element f_3 such that

$$(f_1, f_3) \notin U^q_{0,\dot{\varepsilon}}, \qquad (f_2, f_3) \notin U^q_{0,\dot{\varepsilon}}.$$

Continuing this way, we obtain elements $f_1, f_2, \cdots, f_n$ for which $\rho(f_i, f_j) \notin U^q_{0,\dot{\varepsilon}}$ if $i \neq j$. There exist two possibilities. Either the

1) We put $U^q_{0,\varepsilon} = \{x \in E_1 \mid 0 < x(q) > \varepsilon\}$, $U^q_{0,\dot{\varepsilon}} = \{x \in E_1 \mid 0 < x(q) \leq \varepsilon\}$.