152. *On Arithmetic Properties of Symmetric Functions of Consecutive Integers*

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1. **Main results.** Let n be any integer $\geqslant 2$. We shall write:

$$f_n(x)=\prod_{i=1}^{n}(x+i)=\sum_{k=0}^{\infty}a_k^{(n)}x^k, \tag{1}$$

so that we have:

$$a_0^{(n)}=n!,\quad a_n^{(n)}=1,\quad a_{n+1}^{(n)}=a_{n+2}^{(n)}=\cdots=0$$

and $a_k^{(n)}$ $(1\leq k\leq n-1)$ is the elementary symmetric function of degree $(n-k)$ of n consecutive integers $\{1,2,\cdots,n\}$. These numbers have interesting arithmetic properties as shown in the following theorems:

Theorem 1. *Let p be any prime and suppose $p-1\leq n$. $a_k^{(n)}$ being defined by* (1), *put*

$$b_j^{(n)}=\sum_{\nu=0}^{\infty}a_{j+(p-1)\nu}^{(n)},\quad j=0,1,\cdots,p-2. \tag{2}$$

(*The right-hand side of* (2) *is a finite sum, because* $a_{n+1}^{(n)}=a_{n+2}^{(n)}=\cdots=0$.)

Then we have

$$b_j^{(n)}\equiv 0 \pmod p \tag{3}$$

for $j=0,1,\cdots,p-2$.

Remark. When $p-1=n$, (3) means

$$b_0^{(p-1)}=a_0^{(p-1)}+a_{p-1}^{(p-1)}=(p-1)!+1\equiv 0 \pmod p \tag{4}$$

and

$$a_1^{(p-1)}\equiv a_2^{(p-1)}\equiv\cdots\equiv a_{p-2}^{(p-1)}\equiv 0 \pmod p. \tag{5}$$

(4) is nothing but the classical theorem of Wilson. Thus Theorem 1 can be regarded as a generalization of Wilson's theorem.

From (5) follows, by the fundamental theorem on symmetric functions that any homogeneous symmetric function of $\{1,2,\cdots,p-1\}$ with integral coefficients of a positive degree $\leq p-2$ is always divisible by p. The following theorem gives a more precise result:

Theorem 2. *Let p be any prime $\geqslant 3$. Then any homogeneous symmetric function of $\{1,2,\cdots,p-1\}$ with integral coefficients of odd degree which is $\geqslant 3$ and $\leq p-2$, is always divisible by p^2.*

Some special cases of this theorem are reported in Dickson [1], pp. 95–96.

The following theorem concerns again $a_k^{(n)}$ for general n (not only for $n=p-1$).

Theorem 3. *$a_k^{(n)}$ being defined by* (1) *as above, and p being any*