

195. On Free Abelian m -Groups. III

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In this part, the notion of tensor product of abelian m -groups will be introduced.

Definition. The *tensor product* of the abelian m -groups M and N is defined as F/θ and is denoted by $M \boxtimes N$.

If $|(x, y)|/\theta$ is denoted by $x \boxtimes y$, observe that

$$\begin{aligned} [x_1 x_2 \cdots x_m] \boxtimes y &= [(x_1 \boxtimes y)(x_2 \boxtimes y) \cdots (x_m \boxtimes y)], \\ x \boxtimes [y_1 y_2 \cdots y_m] &= [(x \boxtimes y_1)(x \boxtimes y_2) \cdots (x \boxtimes y_m)], \end{aligned}$$

and $x^{(n)} \boxtimes y = x \boxtimes y^{(n)} = (x \boxtimes y)^{(n)}$.

Theorem 9. Let M, N, P be arbitrary abelian m -groups and $f: M \times N \rightarrow P$ be a function satisfying the conditions

- (a) $f([x_1 x_2 \cdots x_m], y) = [f(x_1, y)f(x_2, y) \cdots f(x_m, y)]$,
- (b) $f(x, [y_1 y_2 \cdots y_m]) = [f(x, y_1)f(x, y_2) \cdots f(x, y_m)]$,
- (c) $f(x^{(n)}, y) = f(x, y^{(n)})$,

for all $x, x_1, \dots, x_m \in M$ and $y, y_1, \dots, y_m \in N$. Then there exists uniquely an m -group homomorphism $h: M \boxtimes N \rightarrow P$ such that the following diagram is commutative

$$\begin{array}{ccc} M \times N & & \\ \boxtimes \downarrow & \searrow f & \\ M \boxtimes N & \xrightarrow{h} & P, \end{array}$$

that is, $h(x \boxtimes y) = f(x, y)$ for all $x \in M$ and $y \in N$.

Proof. Let F be the free abelian m -group on $M \times N$ and $i: M \times N \rightarrow F$ be the injection $i(x, y) = |(x, y)|$. Consider the following diagram.

$$\begin{array}{ccc} M \times N & \xrightarrow{i} & F \\ \boxtimes \downarrow & \searrow f & \downarrow f^* \\ M \boxtimes N & \xrightarrow{h} & P \end{array}$$

(Note: In the original diagram, there is an arrow labeled 'p' from F to M ⊗ N, and an arrow labeled 'f' from M ⊗ N to P. The diagram is a commutative square with an additional arrow from F to P.)

By Theorem 4, f possesses a unique homomorphic extension $f^*: F \rightarrow P$ such that $f^* \cdot i(x, y) = f(x, y)$ so that $f^*(|(x, y)|) = f(x, y)$. Since

$$\begin{aligned} f^*(|[x_1 x_2 \cdots x_m], y|) &= f([x_1 x_2 \cdots x_m], y) \\ &= [f(x_1, y)f(x_2, y) \cdots f(x_m, y)] = [f^*(|(x_1, y)|) \cdots f^*(|(x_m, y)|)], \end{aligned}$$