# 193. *On Free Abelian m-Groups.* I

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In this article, the notions of free abelian $m$-group and the tensor product of abelian $m$-groups will be introduced and their more immediate properties are developed.

Recall that

**Definition.** An algebraic system $(M, [\ \ ])$ or simply $M$ is called an *$m$-semigroup* if and only if $[\ \ ]: M^m \to M$ satisfies the $m$-associative law, i.e.

$[[x_1x_2 \cdots x_m]x_{m+1} \cdots x_{2m-1}] = [x_1x_2 \cdots x_i[x_{i+1}x_{i+2} \cdots x_{i+m}]x_{i+m+1} \cdots x_{2m-1}]$ for each $i=1, 2, \cdots, m-1$ and all $x_1, x_2, \cdots, x_{2m-1} \in M$.

The $m$-ary operation $[\ \ ]$ can be extended in a natural way to an $n$-ary operation, where $n$ is greater than $m$ and such that $n \equiv 1 \pmod{m-1}$. This is done by defining

$$[x_1x_2 \cdots x_n] = [\cdots [[x_1x_2 \cdots x_m]x_{m+1} \cdots x_{2m-1}] \cdots x_n]$$

for all $x_1, x_2, \cdots, x_n \in M$ and $n \equiv 1 \pmod{m-1}$. The following generalized associative law holds for $m$-semigroups (see R. H. Bruck [2]):

$$[x_1x_2 \cdots x_m] = [x_1x_2 \cdots x_i[x_{i+1}x_{i+2} \cdots x_j]x_{j+1} \cdots x_n]$$

for $n \equiv 1 \pmod{m-1}$, $1 < j-i \equiv 1 \pmod{m-1}$, and all $x_1, x_2, \cdots, x_n \in M$.

For convenience, one may designate $\langle k \rangle = k(m-1)+1$ and $x^{\langle k \rangle} = [x_1x_2 \cdots x_{\langle k \rangle}]$ with $x_1 = x_2 = \cdots = x_{\langle k \rangle} = x$. Observe that the following exponential laws hold in any $m$-semigroup: (1) $(x^{\langle h \rangle})^{\langle k \rangle} = x^{\langle hk(m-1)+h+k \rangle}$ and (2) $[x^{\langle k_1 \rangle}x^{\langle k_2 \rangle} \cdots x^{\langle k_m \rangle}] = x^{\langle k_1+k_2+\cdots+k_m+1 \rangle}$.

**Definition.** An $(m-1)$-tuple $(u_1, u_2, \cdots, u_{m-1})$ of elements from an $m$-semigroup $(M, [\ \ ])$ is called an *$(m-1)$-adic identity* of $M$ if and only if $[xu_1u_2 \cdots u_{m-1}] = x = [u_1u_2 \cdots u_{m-1}x]$ for all $x \in M$. In a similar manner, for any $n \equiv 1 \pmod{m-1}$, the notion of a $(n-1)$-adic identity of $M$ may be defined.

Note that $(u_1, u_2, \cdots, u_{k(m-1)})$ is a $k(m-1)$-adic identity if and only if $([u_1u_2 \cdots u_{(k-1)(m-1)}u_{(k-1)(m-1)+1}], \cdots, u_{k(m-1)})$ is an $(m-1)$-adic identity.

**Definition.** An $m$-semigroup $(M, [\ \ ])$ is an *$m$-group* if and only if

(a) for $u_1, u_2, \cdots u_{m-2} \in M$, there exists a $u \in M$ such that $(u_1, u_2, \cdots, u_{m-2}, u)$ is an $(m-1)$-adic identity of $M$;

(b) for $u_1, u_2, \cdots u_{m-2} \in M$, there exists a $u \in M$ such that $(u, u_1, \cdots, u_{m-2})$ is an $(m-1)$-adic identity of $M$.