191. Some Properties of M-Spaces

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In a previous paper [9] we have introduced the notion of M-spaces. A topological space X is an M-space if there exists a normal sequence $\{\mathfrak{U}_i \mid i=1, 2, \cdots\}$ of open coverings of X satisfying condition (M) below:

$$(M)\quad \begin{cases} \text{If } \{K_i\} \text{ is a sequence of non-empty subsets of } X \text{ such that} \\ K_{i+1}\subset K_i,\ K_i\subset \mathrm{St}(x_0, \mathfrak{U}_i) \text{ for each } i \text{ and for some fixed point} \\ x_0 \text{ of } X, \text{ then } \cap \bar{K}_i\neq\phi. \end{cases}$$

Condition (M) is equivalent to the original condition (40) in [9]. In this paper we shall discuss some properties of M-spaces.

1. In [7] (cf. also J. Dugundji [2, p. 196]) we proved that a T_1-space X is metrizable if and only if there is a sequence $\{\mathfrak{F}_i\}$ of locally finite closed coverings of X such that for any point x and for any neighborhood V of x there is some i for which $\mathrm{St}(x, \mathfrak{F}_i)\subset V$. Thus it is natural to consider a topological space X such that there is a sequence $\{\mathfrak{F}_i\}$ of locally finite closed coverings of X satisfying condition (M). Such a space we shall call an M^*-space after T. Ishii [3]. Corresponding to our metrization theorem mentioned above, the following theorem holds.

Theorem 1.1. *If X is an M-space, then X is an M^*-space with property (C) below:*

$$(C)\quad \begin{cases} \text{For any locally finite collection } \{F_\lambda\} \text{ of closed sets of } X \text{ there} \\ \text{exists a locally finite collection } \{G_\lambda\} \text{ of open sets of } X \text{ such} \\ \text{that } F_\lambda\subset G_\lambda \text{ for each } \lambda. \end{cases}$$

In case X is normal, the converse is true.

Proof. Let X be an M-space; by [8, Theorem 1. 2] there is a normal sequence $\{\mathfrak{V}_i\}$ of locally finite open coverings of X satisfying condition (M), and hence $\{\mathfrak{F}_i\}$ satisfies condition (M) where we set $\mathfrak{F}_i=\{\bar{V}\mid V\in\mathfrak{V}_i\}$. Since by A. Okuyama [10] any M-space has property (C), the first part of the theorem is proved.

To prove the second part, let $\{\mathfrak{F}_i\}$ be a sequence of locally finite closed coverings of X satisfying condition (M); without loss of generality we may assume that $\mathfrak{F}_{i+1}$ is a refinement of $\mathfrak{F}_i$ for $i=1, 2, \cdots$. Let us set

$$(1)\qquad \mathfrak{L}_i=\{\mathrm{St}(F, \mathfrak{F}_i)\mid F\in\mathfrak{F}_i\},\qquad i=1, 2, \cdots.$$

Then, if $A\subset X$, we have