182. On the Spherical Derivative of Functions Regular or Meromorphic in the Unit Disc

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1. Introduction. O. Lehto and K. Virtanen [3] used the spherical derivative

$$\rho(f(z))=\frac{|f'(z)|}{1+|f(z)|^2} \tag{1.1}$$

as a measure of the growth of $f(z)$ near an isolated singularity, and they [1, 2] developed the study of this direction. In particular, as regards the growth of the spherical derivative Lehto proved:

Theorem A. *Let $f(z)$ be meromorphic in a neighbourhood of the essential singularity $z=a$. Then*

$$\overline{\lim_{z\to a}}\, |z-a|\, \rho(f(z)) \geqq \frac{1}{2}. \tag{1.2}$$

Equality holds for the product

$$f(z)=\prod_{\nu} \frac{z-a-a_\nu}{z-a+a_\nu},$$

where the numbers a_ν satisfy the condition $|a_{\nu+1}|=o(|a_\nu|)$.

Theorem B. *If $f(z)$ satisfies the hypothesis of Theorem A and further $f(z)$ is regular near $z=a$, then*

$$\overline{\lim_{z\to a}}\, |z-a|\, \rho(f(z))=\infty. \tag{1.3}$$

Further J. Clunie and W. K. Hayman obtained some extensions of Theorem A and B in their paper [4]. For instance, they proved the following result.

Theorem C. *If $f(z)$ is an integral function of proper order λ $(0\leqq\lambda\leqq\infty)$, then*

$$\overline{\lim_{r\to\infty}} \frac{r\mu(r, f)}{\log M(r, f)} \geqq A_0(\lambda+1), \tag{1.4}$$

where A_0 is an absolute constant and $\mu(r, f)=\sup_{|z|=r} \rho(f(z))$.

2. Our object in this paper is to obtain some results concerning the growth of spherical derivative $\rho(f(z))$ for functions regular and meromorphic in the unit disc $|z|<1$. First we shall prove:

Theorem 1. *Suppose that $f(z)$ is regular for $|z|<1$ and that its order λ satisfies $2<\lambda\leqq\infty$. Then*

$$\overline{\lim_{r\to 1}}\, (1-r)^{\lambda-1}\mu(r, f) \geqq K\lambda\left(\frac{\lambda-2}{\lambda+2}\right)^{\lambda-1} \tag{2.1}$$