181. On the Analyticity and the Unique Continuation Theorem for Solutions of the Navier-Stokes Equation

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1. Introduction. Consider the Navier-Stokes equation

(1) $\boldsymbol{u}_t+(\boldsymbol{u}\cdot\mathrm{grad})\boldsymbol{u}=\Delta\boldsymbol{u}-\nabla p+\boldsymbol{f},\ \mathrm{div}\,\boldsymbol{u}=0,\ x\in G,\ 0<t<T,$

and the condition of adherence at the boundary

(2) $\boldsymbol{u}=0$ on the boundary of G.

Here G is a connected component of exteriors (or interiors) of a bounded hypersurface of class C^2, $\boldsymbol{u}$ and $\boldsymbol{f}$ are 3-dimensional real vector functions of x and t, and p is a scalar function of x and t. We are mainly concerned with the question whether a nonconstant flow of incompressible fluid, subject to the Navier-Stokes equation (1) with $f=0$ and the condition (2) of adherence at the boundary, can ever come to rest in a finite time on some potion of G. Before stating our results, we shall define function spaces, and fix our notations. For any open set Q in R^n, $\boldsymbol{W}^{k,p}(Q)$ $(k\geqslant 0, 1\leqslant p<\infty)$ is the set of all complex-valued vector functions in $\boldsymbol{L}^p(Q)$ for which distribution derivatives of up to order k lie in $\boldsymbol{L}^p(Q)$. $\boldsymbol{W}^{k,p}(Q)$ $(k>0)$ is the set of all distributions $\boldsymbol{u}$ such that $|\langle\boldsymbol{u},\boldsymbol{\varphi}\rangle^{1)}|\leqslant C\,\|\boldsymbol{\varphi}\|_{L^p}$ for $\boldsymbol{\varphi}$ in $C_0^\infty(Q)$, C being a positive constant, where $\|\boldsymbol{\varphi}\|_{L^p}$ is the L^p-norm of $\boldsymbol{\varphi}$. $\boldsymbol{W}^{k,p}_{\mathrm{loc}}(Q)$ $(k=0,\pm 1,\cdots)$ is the set of all distribution $\boldsymbol{u}$ on Q which coincide on some neighborhood of each point of Q with elements of $\boldsymbol{W}^{k,p}(Q)$. The set of all 3-dimensional real vector functions φ such that $\varphi\in C_0^\infty(G)$, and $\mathrm{div}\,\varphi=0$, is denoted by $C_{0,s}^\infty(G)$. Let $\boldsymbol{L}_s^2=\boldsymbol{L}_s^2(G)$ be the closure of $C_{0\ s}^\infty(G)$ in $\boldsymbol{L}^2(G)$. Let P be the orthogonal projection from $\boldsymbol{L}^2(G)$ onto $\boldsymbol{L}_s^2$. By A we denote the Friedrichs extension of the symmetric operator $-P\Delta$ in $\boldsymbol{L}_s^2$ defined for every $\boldsymbol{u}$ such that $\boldsymbol{u}\in C^2(G)\cap C^1(G^a)$, $\mathrm{div}\,\boldsymbol{u}=0$, and $\boldsymbol{u}=0$ on the boundary of G, G^a being the closure of G. By X_γ we denote the set of all $\boldsymbol{u}$ in $D(A^\gamma)$ with the norm $\|\boldsymbol{u}\|_{X_\gamma}=\|A^\gamma\boldsymbol{u}\|+\|\boldsymbol{u}\|$, $D(A^\gamma)$ being the domain of A^γ, where γ is any number with $3/4<\gamma<1$. We let $\boldsymbol{X}=\boldsymbol{X}_{4/5}$. Here $\|\cdot\|$ is the norm of the Hilbert space $\boldsymbol{L}^2(G)$ with the scalar product $(\cdot,\cdot)$. Let $\boldsymbol{H}_{0,s}^1=\boldsymbol{H}_{0,s}^1(G)$ be the completion of the set $C_{0,s}^1(G)$ of all solenoidal $(\mathrm{div}\,\boldsymbol{u}=0)$ functions in C_0^1 with the norm $\|\nabla\boldsymbol{u}\|+\|\boldsymbol{u}\|$. Now our results are as follows.

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1) $\langle\boldsymbol{u},\boldsymbol{\varphi}\rangle$ denotes the value of the functional $\boldsymbol{u}$ at $\boldsymbol{\varphi}$.