5. *A Remark on the Contraction Principle*

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In his paper [1], E. Dubinsky states the following fixed point theorem:

Let X_0 be an open neighborhood of 0 in a complete locally convex space E, and let f be a mapping of x_0+X_0, where $x_0 \in E$, into E satisfying the condition that there exist a non-empty closed bounded convex subset B of X_0 and a non-negative real number $k<1$ such that

$$x, y \in x_0+X_0 \text{ and } x-y \in \lambda B \text{ imply } f(x)-f(y) \in \lambda k B.$$

Then if $f(x_0)-x_0 \in (1-k)B, f$ has a unique fixed point in x_0+B.

The proof is, in a sense, analogous to that of the well-known Banach contraction theorem, and so it will be natural to ask the relation between these two theorems. The purpose of this note is to clarify the positions of these theorems. That is we shall state a basic theorem (Theorem 1 below) from which these theorems follow, and we shall give a slight generalization of the theorem of Dubinsky (Theorem 2).

The vector spaces we shall be concerned with in this note are over the real number field R or the complex number field. We employ the following notations: $[0, \alpha] \equiv \{\xi \in R; 0 \leq \xi \leq \alpha\}$ and $[0, \alpha) \equiv \{\xi \in R; 0 \leq \xi < \alpha\}$ where α is a positive real number.

1. A triple $\langle X, D, d\rangle$ of a set X, a subset D of $X \times X$ and a non-negative real valued function d defined on D is called a *premetric space* (and d a *premetric* for X with domain D) if the following two conditions are satisfied:

(P 1) For every $x \in X, (x, x) \in D$, and $d(x, x)=0$.

(P 2) If $(x, y), (y, z) \in D$, then $(x, z) \in D$ and

$$d(x, z) \leq d(x, y)+d(y, z).$$

Let $\langle X, D, d\rangle$ be a premetric space. If M is a subset of X, then $\langle M, D\cap (M\times M), d|_{D\cap(M\times M)}\rangle$ is also a premetric space, where $d|_{D\cap(M\times M)}$ denotes the restriction of d to $D\cap(M\times M)$; we shall call it a *subspace* of $\langle X, D, d\rangle$ and denote simply by M.

If d is a premetric for a set X with domain D, then by setting $d^*(x, y)=d(y, x)$ for every $(y, x) \in D$, a premetric d^* for X with domain $\{(x, y); (y, x) \in D\}$ is obtained; we shall call d^* the *dual premetric* of d.

A sequence $\{x_n\}$ in a premetric space $\langle X, D, d\rangle$ is *r-convergent* to $x \in X$ if $(x, x_n) \in D$ for every n, and if there exists, for each $\varepsilon<0$, a positive integer n_0 such that $d(x, x_n)<\varepsilon$ whenever $n \geq n_0$. A