11. *An Ergodic Theorem*

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Let A be a bounded linear operator in a Banach space X with uniformly bounded powers:

(1) $$\|A^n\| \leqq M < \infty, \qquad n=1, 2, \cdots.$$

If X is reflexive, then the mean ergodic theorem of Yosida [4] and Kakutani [2] asserts that the mean of powers $\sum_{k=1}^{n} A^k/n$ converges strongly to a projection onto the eigen-space $N(1-A)=\{x \in X; (1-A)x=0\}$. Hille [1] showed also that the arithmetic mean may be replaced by any Cesaro mean of positive order. This seems to be the farthest one can expect under condition (1), for, obviously there are operators A with (1) such that A^n does not converge in any sense.

The purpose of this paper is to prove the following theorem which gives a sufficient condition in order that A^n converge strongly.

Assumptions. A is a bounded linear operator in a Banach space X which satisfies the following:

(a) The spectrum $\sigma(A)$ is contained in the unit disc $|\lambda| \leqq 1$;

(b) 1 is the only spectrum on the unit circle $|\lambda|=1$;

(c) There is an angle θ with $0 \leqq \theta < \pi/2$ such that the sector $S=\{\lambda \in \boldsymbol{C}; |\arg(\lambda-1)| < \pi-\theta\}$ is contained in the resolvent set $\rho(A)$ and

(2) $$\|(\lambda-1)(\lambda-A)^{-1}\| \leqq K < \infty, \qquad \lambda \in S.$$

Conclusions. (i) A satisfies (1);

(ii) $N(1-A)=N((1-A)^m)$, $m=1, 2, \cdots$;

(iii) The closure $\overline{R(1-A)}$ of the range $R(1-A)=\{(1-A)x; x \in X\}$ and $N(1-A)$ have only zero as the common elements;

(iv) $N(1-A)+\overline{R(1-A)}$ is a closed subspace of X, and coincides with X if X is reflexive;

(v) If $x=x_0+x_1$ with $x_0 \in N(1-A)$ and $x_1 \in \overline{R(1-A)}$, then $A^n x$ converges strongly to x_0 as $n \to \infty$;

(vi) Let $x \in X$. If there is a sequence $n_j \to \infty$ such that $A^{n_j}x$ converges weakly, then x belongs to $N(1-A)+\overline{R(1-A)}$.

Proof of (i). If Γ is a path which encircles $\sigma(A)$, we have

$$A^n = \frac{1}{2\pi i}\int_{\Gamma} \zeta^n (\zeta - A)^{-1} d\zeta.$$

Let Γ be the union of $\Gamma_1, \Gamma_2, \Gamma_3$, and Γ_4, where Γ_1 is the segment that connects $1+\cos\theta e^{i(\theta-\pi)}$ and $1+n^{-1}e^{i(\theta-\pi)}$, Γ_2 is the part of circle