36. *On an Analytic Index-formula for Elliptic Operators*

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§ 1. **Preliminaries.** In his work [1], [3], M. F. Atiyah indicated an analytic formula for the index of elliptic differential operators on compact manifolds. The aim of this note is to describe this formula more explicitly.

Assume that both X and Y are differentiable vector bundles with fibre C^l over a compact oriented Riemannian manifold M without boundary and that they are provided with hermitian metric in each fibre. Let P be an elliptic differential operator of order m from $\mathcal{E}(X)$ to $\mathcal{E}(Y)$, where $\mathcal{E}(X)$ is the space of C^∞ sections of X provided with the usual topology. We denote by $L^2(X)$ the space of L^2 sections of X. Then, considered as a densely defined linear operator from $L^2(X)$ to $L^2(Y)$, P is closable. We denote its minimal closed extension by the same symbol P. Since P is a densely defined closed operator, there is its adjoint P^* which is a densely defined closed operator from $L^2(Y)$ to $L^2(X)$. It is well known that P has a finite index $\operatorname{Ind}(P)$.

§ 2. **Results.** Our first result is the following:

Theorem 1. *Let λ be a positive number. Then we have the formula*

$$(1)\qquad \operatorname{Ind}(P)=\lim_{\lambda\to\infty}\lambda[\operatorname{Trace}(\lambda+(P^*P)^k)^{-1}-\operatorname{Trace}(\lambda+(PP^*)^{k-1}]$$

where k is an arbitrary integer which is larger than $\dfrac{n}{2m}$.

Proof. The following proof is a variant of the discussion used in M. F. Atiyah and R. Bott [3].

Let $\Lambda=\{0,\lambda_1,\lambda_2,\cdots\}$ be the set of eigen values of PP^* or P^*P with $0<\lambda_1<\lambda_2<\cdots$. Let $\Gamma_j(X)$ and $\Gamma_j(Y)$ be, respectively, the eigen-spaces of P^*P and PP^* corresponding to λ_j. It is well known that $\Gamma_j(X),\Gamma_j(Y)$ are of finite dimension. Let P_j denote the restriction of P to $\Gamma_j(X)$. Then we have the following complexes:

$$0\longrightarrow\Gamma_j(X)\xrightarrow{P_j}\Gamma_j(Y)\longrightarrow 0,\qquad j=0,1,2,3,\cdots.$$

Obviously,

$$\operatorname{Ind}(P)=\dim\Gamma_0(X)-\dim\Gamma_0(Y),$$
$$0=\dim\ker P_j-\dim\operatorname{coker}P_j,$$

because $P^*P|_{\Gamma_j(X)}=\lambda_j$, $PP^*|_{\Gamma_j(Y)}=\lambda_j$. Hence