28. On Automorphisms of an Injective Module

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(Comm. by Kenjiro SHODA, M. J. A., March 12, 1968)

1. Statement of the main result. Throughout this paper we assume that every ring has an identity element and an R-module means a unital left R-module. Let $B=\mathrm{Hom}_R(M, M)$ be an R-endomorphism ring of an R-module M as a right operator domain of M. In this paper we shall be concerned with the following condition:

Condition (0). $Me\approx M$, $e=e^2\in B$, implies $e=1$.

It is easy to see that if any isomorphism between two R-submodules of M can be extended to an automorphism of M, then M satisfies Condition (0). Our aim is to prove the following theorem.

Theorem 1. *Let M be an injective R-module with Condition (0). Then any isomorphism between two R-submodules of M can be extended to an automorphism of M.*

2. Left self-injective, regular rings with Condition (0). We denote the injective envelope [1] of an R-module A by $E(A)$. We write $N'{}'\supset N$ if N' is an essential extension of N. If X is a subset of a ring S, we define the left (resp. right) annihilator

$$l(X)=\{s\in S \mid sX=0\}$$

(resp. $r(X)$, similarly). We shall list a series of lemmas.

Lemma 2. *Let S be a left self-injective, regular ring. Then every left annihilator ideal A is generated by an idempotent.*

Proof. By the regularity of S, we have $r(A)=\bigcup_{e=e^2\in r(A)} eS$. Then

$$A=l(r(A))=l(\bigcup_{e\in r(A)} eS)=\bigcap_{e\in r(A)} l(eS)=\bigcap_{S(1-e)\supset A} S(1-e).$$

But, for each $S(1-e)\supset A$, $E(A)'\supset S(1-e)\cap E(A)'\supset A$ and hence $E(A)=S(1-e)\cap E(A)\subset S(1-e)$ by the injectivity of $S(1-e)\cap E(A)$. Therefore $A=E(A)=Sf$ for some $f=f^2\in S$.

Lemma 3. (J. von Neumann [7, Lemma 18]). *Let S be a regular ring. Then a principal left ideal of S is a two-sided ideal if and only if it is generated by a central idempotent.*

Lemma 4. (B. Eckmann and A. Schopf [1, 4.3]). *Let $v: A\to A'$ be an R-isomorphism, then v can be extended to an R-isomorphism of $E(A)$ onto $E(A')$.*

Lemma 5. *For any two idempotents e, f of a regular ring S, the following conditions are equivalent:*

(1) $eSf\neq 0$.

(2) $Se'\approx Sf'$ *for some* $0\neq Se'\subset Se$ *and* $Sf'\subset Sf$.