24. *On the Application of the Potential Theory to Martingales*

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Introduction. In his recent book [1], P. A. Meyer mentioned a remark on the mapping of a set of processes into itself stated below, which enables one to apply the theory of potentials to that of martingales:

Let $(\Omega, \boldsymbol{F}, \boldsymbol{P})$ be a probability space, and $\{\boldsymbol{F}_n\}_{n\in \boldsymbol{N}^+}$ be an increasing family of sub-σ-fields of $\boldsymbol{F}$, where $\boldsymbol{N}^+=\{1, 2, 3, \cdots\}$. Put $\boldsymbol{S}=\Omega\times\boldsymbol{N}^+$, the product space, and attach to it the σ-field $\mathcal{F}$ consisting of the sets of the form $\bigcup_{n=1}^{\infty} A_n\times\{n\}$, where $A_n\in\boldsymbol{F}_n$. Then any real-valued $\mathcal{F}$-measurable function may be identified with a process adapted to $\{\boldsymbol{F}_n\}_{n\in\boldsymbol{N}^+}$. If we denote by $\mathcal{A}$ the family of the sets of the form $\bigcup_{n=1}^{\infty} A_n\times\{n\}$, where $A_n\in\boldsymbol{F}_n$ and $\boldsymbol{P}(A_n)=0$, then $\mathcal{A}$ is closed under countable union. We can define the mapping N of a certain class of processes into itself in the following manner:

$$(NX)_n=\boldsymbol{E}(X_{n+1}\,|\,\boldsymbol{F}_n).$$

N determines the process with the ambiguity of the values on the sets belonging to $\mathcal{A}$.

In this paper, we define a kernel which is a generalization of the usual kernel, establish the potential theory associated with the kernel, and deduce some theorems on martingales, though mostly already known, using above notions and the method suggested in Doob's paper [2].

§1. Sub-Markov pseudo kernels and potential theory. Let $\boldsymbol{S}$ be an abstract space and $\mathcal{F}$ be a σ-field of subsets of $\boldsymbol{S}$. Let $\mathcal{A}$ be a subfamily of $\mathcal{F}$ closed under the operation of countable union. We denote by $\mathcal{P}^0$ the set of all $\mathcal{F}$-measurable functions on $\boldsymbol{S}$ with values in $[0, +\infty]$, and define the equivalence relation $\sim$ in $\mathcal{P}^0$ as follows:

$f\sim g$ if and only if $f(s)=g(s)$ on $\boldsymbol{S}-A$ for some $A\in\mathcal{A}$.

We classify $\mathcal{P}^0$ by this equivalence relation and set $\mathcal{P}=\mathcal{P}^0/\sim$. Then we can naturally define the usual algebraic operations and limit processes in $\mathcal{P}$ from the corresponding operations in $\mathcal{P}^0$; this may be done by the same way as we do for function spaces on a measure space in case $\mathcal{A}$ is the totality of sets of measure zero.