58. Compactness and Completeness in Ranked Spaces

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The purpose of this note is to study the relation between the compactness and the completeness in ranked spaces.

The notion of the completeness in the ranked spaces was introduced in the note [1], and that of the compactness in the notes [3] and [6]. Every complete metric space can be considered a complete ranked space.

In any metric space, every sequentially compact set is complete. This assertion does not always hold in ranked spaces. In the first half of this note, we shall show it and give some condition which makes the assertion hold. And in the latter, we shall give some conditions with which the completeness yields the compactness. Throughout this note, we shall always treat ranked spaces with indicator ω_0 ([2], p. 319), and $m, n, \cdots$ will denote non-negative integers.

§ 1. From the compactness. A sequence $\{U_n(x_n)\}_{n=0,1,2,\dots}$ of subsets of a ranked space is called a *fundamental sequence* if it possesses the following three properties ([7] p. 1142):

(1) any $U_n(x_n)$ is a neighbourhood of point x_n of rank r_n, and

$$r_0 \leq r_1 \leq r_2 \leq \cdots \leq r_n \leq \cdots, \text{ and } \lim_{n\to\infty} r_n = +\infty;$$

(2) $U_0(x_0) \supseteq U_1(x_1) \supseteq U_2(x_2) \supseteq \cdots \supseteq U_n(x_n) \supseteq \cdots$;

(3) for any n, there is an m such that $r_m < r_{m+1}$, and that a neighbourhood $V(x_m)$ of point x_m of rank s $(r_m < s \leq r_{m+1})$ containing $U_{m+1}(x_{m+1})$ and included in $U_m(x_m)$ exists:

$$U_m(x_m) \supseteq V(x_m) \supseteq U_{m+1}(x_{m+1}).$$

A ranked space is said to be *complete* if, for every fundamental sequence $\{U_n(x_n)\}$, we have $\bigcap_n U_n(x_n) \neq \phi$.

Example 1. Let E be the interval $[0, 1]$ of real numbers and, for any x of E and for any n, let

$$\mathfrak{B}_{2n}(x) = \mathfrak{B}_{2n+1}(x) = \left\{ \left(x - \frac{1}{n},\ x + \frac{1}{n} \right) \cap E \right\}.^{1)}$$

Then E is a ranked space, and is r-compact ([6]). But it is not complete, because for the fundamental sequence $\{U_n(x_n)\}_{n=0,1,2,\dots}$ where

1) $\mathfrak{B}_n(x)$ will denote the family of neighbourhoods of point x and of rank n. [8] p. 616.