79. On Banach Function Spaces

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The theory of Riesz spaces (i.e. a normed vector lattice) plays an important role in the theory of normed function spaces. The theory have been developed by W. A. J. Luxemburg and A. C. Zaanen (see [1], [2]).

First I explain some terminologies (see [2]). Let X be a non-empty set and μ a non-negative, countable additive measure on X. We denote by $(X, \sum, \mu)$ a σ-finite measure space. Let M be the set of all real valued, μ-measurable functions on X, and M^+ the set of all non-negative functions of M. A function seminorm ρ is a mapping of M^+ into the real numbers and has the seminorm properties and $\rho(u)\leqslant\rho(v)$ if $u(x)\leqslant v(x)$ almost everywhere on X. We extend the domain of ρ to the whole M by defining $\rho(f)=\rho(|f|)$. The normed function space L_ρ is the set of $f\in M$ such that $\rho(f)<\infty$. We assume that there is at least one $f\in M$ such that $0<\rho(f)<\infty$. We introduce two function seminorms ρ_1, ρ_2 as follows

$$\rho_1(f)=\operatorname*{Sup}_{\rho(g)\leq 1}\left\{\int|fg|d\mu\right\},\qquad \rho_2(f)=\operatorname*{Sup}_{\rho_1(g)\leq 1}\left\{\int|fg|d\mu\right\}.$$

A measurable subset B of X is called *ρ-purely infinite,* if $\rho(\chi_c)=\infty$ for every $C(\subset B)$ of positive measure. ρ is called a *saturated function seminorm* if there is no ρ-purely infinite subsets. There is no loss of generality even if we remove the maximal ρ, ρ_1-purely infinite sets X_∞, X'_∞ from X (see Theorem 12. 1 in [2]). Then ρ, ρ_1, ρ_2 become the saturated function norms. We only use saturated function norms. Under this assumption, there is a sequence (π); $X_n\uparrow X$ such that $0<\mu(X_n)<\infty$ and $0<\rho(\chi_{X_n})<\infty$ (see Theorem 8.7 in [2]). We call such a sequence $(\pi): X_n\uparrow X$ a *ρ-exhaustive sequence.* We introduce the partial ordering in L_ρ by the following way: $f\leq g$ if and only if $f(x)\leq g(x)$ almost everywhere on X. Then L_ρ is a Riesz space with respect to the above ordering. Futher every nonempty subset of L_ρ which is bounded from above has a least upper bound in L_ρ, and it can be obtained by picking out an appropriate increasing subsequence. Such a Riesz space is called *super Dedekind complete.*

Let L_ρ^* be the Banach dual of L_ρ, and $L_{\rho,c}^*$ the subset of L_ρ^* having the following property; $F(\in L_\rho^*)$ belongs to $L_{\rho,c}^*$ if and only if $|f_n(x)|\downarrow 0$ (a.e) implies $F(|f_n|)\to 0$.

We shall now define two subsets of L_ρ as follows.