75. On the Standard Complexes of Cotriples

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In this paper we show that the standard complexes of cotriples are acyclic in most cases. This generalizes Proposition 4.1 of Eilenberg and Moore [4]. For example we confirm that the ordinary (co)homology of modules over an arbitrary ring (Cartan and Eilenberg [2] Chap. VI) is a cotriple (co)homology.

We follow the notions and notations of Eilenberg and Moore [3], [4]. Let $\mathcal{A}$ be a pre-additive category with kernels, $\boldsymbol{G}=(G, \varepsilon, \delta)$ be a cotriple in $\mathcal{A}$. Then $\{G^{n+1}\}_{n\geqq 0}$ is a simplicial object with the face operators $\varepsilon^i=G^i\varepsilon G^{n-i}: G^{n+1}\longrightarrow G^n$ and the degeneracy operators $\delta^i=G^i\delta G^{n-i}: G^{n+1}\longrightarrow G^{n+2}$. The *standard complex* of the cotriple $\boldsymbol{G}$ is defined by the sequence

$$(1)\qquad \cdots\longrightarrow G^{n+1}\xrightarrow{\partial_n} G^n\longrightarrow\cdots\xrightarrow{\partial_1} G\xrightarrow{\varepsilon} 1_{\mathcal{A}}\longrightarrow 0$$

where $\partial_n=\sum_{i=0}^n(-1)^i\varepsilon^i$. Let $\mathcal{G}$ be a projective class of sequences in $\mathcal{A}$ such that $\mathcal{G}$-projective objects are the objects $G(A)$, $A\in ob\mathcal{A}$ and their retracts.

Theorem. *In the above situation, the sequence*

$$(2)\qquad \cdots\longrightarrow G^{n+1}(A)\xrightarrow{\partial_n} G^n(A)\longrightarrow\cdots\xrightarrow{\partial_1} G(A)\xrightarrow{\varepsilon} A\longrightarrow 0$$

is a $\mathcal{G}$-projective resolution of A for any object A in $\mathcal{A}$.

Proof. Every cotriple is generated by an adjoint pair of functors ([4] Theorem 2.2 or [5]):

$$(3)\qquad (\varepsilon, \eta): S\dashv T: (\mathcal{A}, \mathcal{B})$$

i.e. $\boldsymbol{G}=(G, \varepsilon, \delta)$ is represented by $(ST, \varepsilon, S\eta T)$. We may suppose that $\mathcal{B}$, S, T are pointed. Since $\varepsilon S\cdot S\eta=1_S$, an object $A\in ob\mathcal{A}$ is a retract of $G(A')=ST(A')$ for some $A'\in ob\mathcal{A}$ if and only if A is a retract of $S(B)$ for some $B\in ob\mathcal{B}$. Hence the isomorphism of functors

$$(4)\qquad \mathcal{A}(S,\ )\longrightarrow\mathcal{B}(\ ,T)$$

implies $\mathcal{G}=T^{-1}\mathcal{E}_0$ where $\mathcal{E}_0$ is the class of all split exact sequences in $\mathcal{B}$. To prove the theorem we may show that the sequence

$$(5)\qquad \cdots\longrightarrow TG^{n+1}(A)\xrightarrow{T\partial_n} TG^n(A)\longrightarrow\cdots\xrightarrow{T\partial_1} TG(A)\xrightarrow{T\varepsilon} T(A)\longrightarrow 0$$

is split exact for every $A\in ob\mathcal{G}$.

Define morphisms $t_n: G^m\longrightarrow G^m$ and $u_n: G^m\longrightarrow G^{m+1}$, $n<m$, as follows

$$t_n=(1-\delta^0\varepsilon^1)(1-\delta^1\varepsilon^2)\cdots(1-\delta^{n-1}\varepsilon^n),$$