71. *Calculus in Ranked Vector Spaces. IV*

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1.9. The special case. (1.9.1) Proposition. *Let E be a normed vector space, $\{x_n\}$ a sequence of E and $x \in E$. Then for a sequence $\{x_n\}$ converges to x in the sense of ranked vector space it is necessary and sufficient that it converges to x in the sense of norm, i.e.,*

$$\{\lim x_n\} \ni x \Longleftrightarrow \lim \|x_n - x\| = 0.$$

Proof. (a) Suppose that $\{\lim x_n\} \ni x$, i.e., there exists a sequence $\{U_n(x)\}$ of neighborhoods of the point x and a sequence $\{\alpha_n\}$ of integers such that,

$$U_0(x) \supset U_1(x) \supset U_2(x) \supset \cdots \supset U_n(x) \supset \cdots, \quad 0 \leq n < \omega_0,$$
$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq \cdots, \quad 0 \leq n < \omega_0,$$
$$\sup_n \alpha_n = \omega_0, \; U_n(x) \ni x_n, \text{ and } U_n(x) \in \mathfrak{V}_{\alpha_n},$$

for $n = 0, 1, 2, \cdots$.

By (1.6.6), each $U_n(x)$ is written in the following form, using $U_n(x) \in \mathfrak{V}_{\alpha_n}$,

$$U_n(x) = x + V_{\alpha_n}(0), \qquad n = 0, 1, 2, \cdots$$

where $V_{\alpha_n}(0) = \left\{x\,;\ \|x\| < \dfrac{1}{\alpha_n}\right\}$.

For every $\varepsilon > 0$, there exists a positive number N, using $\sup \alpha_n = \omega_0$, such that

$$n \geq N \Rightarrow \frac{1}{\alpha_n} < \varepsilon.$$

Since $U_n(x) = x + V_{\alpha_n}(0) \ni x_n$, $V_{\alpha_n}(0) \ni x_n - x$

$$\therefore \quad \|x_n - x\| < \frac{1}{\alpha_n}.$$

Thus if $n \geq N$, then

$$\|x_n - x\| < \frac{1}{\alpha_n} < \varepsilon$$
$$\therefore \quad \lim \|x_n - x\| = 0.$$

(b) Suppose coversely that $\lim \|x_n - x\| = 0$, then, for 1, there exists a positive number n_1 such that

$$n \geq n_1 \Rightarrow \|x_n - x\| < 1,$$
$$\therefore \quad V_1(0) \ni x_{n_1} - x,\ x_{n_1+1} - x,\ \cdots,\ x_{n_1+i} - x,\ \cdots$$