98. Generalizations of the Alaoglu Theorem with Applications to Approximation Theory. I*)

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1. Introduction. This paper concerns the approximation of set-valued maps by multilinear maps on direct sums of normed linear spaces. In particular, we prove two very general existence theorems for best approximations in the sense of Tchebycheff (Theorems 3 and 7). One of these theorems (Theorem 7) generalizes one in [1, p. 97] and can be interpreted as a generalization also of the Alaoglu theorem [2, p. 424]. We recall that a real-valued function defined on a topological space X is called *lower-semicontinuous* if each set of the form $\{x \in X : f(x) \leq c\}$ is closed. The fact that a lower-semicontinuous function defined on a compact topological space is bounded below and achieves its infinum thereon will be needed frequently, and will be used freely without explicit reference.

2. Definitions. Let E and F be normed linear spaces over the same scalar field. Let Q be a set-valued map of $X(X \subseteq E)$ into F. We use the symbol $\|Q\|_X$ to denote the number $\sup\{\|p\| : p \in Qx$ for some $x \in X\}$. We say that Q is *bounded* if $\|Q\|_X$ is finite. Let K be another set-valued map of X into F. We define $K-Q$ in the most natural way, that is $(K-Q)(x)=Kx-Qx=\{p-q : p \in Kx$ and $q \in Qx\}$. Let M be a family of set-valued maps of X into F. A member P_0 of M is termed a *best approximation in M to Q* if $\|P_0-Q\|_X=\inf\{\|P-Q\|_X : P \in M\}$.

Let $E_1, \cdots, E_n, F$ be normed linear spaces over the same scalar field. The direct sum of the spaces $E_1, \cdots, E_n$ is the normed linear space of all ordered n-tuples $[x_1, \cdots, x_n]$, where $x_i \in E_i$, $i=1, \cdots, n$, with component wise addition and component wise scalar multication, and with the norm defined by $\|[x_1, \cdots, x_n]\|=\max\{\|x_i\| E_i : i=1, \cdots, n\}$. We denote this direct sum by the symbol $E_1 \oplus \cdots \oplus E_n = E$.

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