96. Calculus in Ranked Vector Spaces. VI

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3.4. Differentiable mappings into a direct product.

(3.4.1) **Proposition.** *Let $F_1, F_2, \cdots, F_m$; $E_1, E_2, \cdots, E_m$ be a family of separated ranked vector spaces and $f_1: F_1 \to E_1, f_2: F_2 \to E_2, \cdots, f_m: F_m \to E_m$ a family of mappings. Let $\times f_i: \times F_i \to \times E_i$ be a mapping from $\times F_i$ into $\times E_i$ defined by,*

$$(\times f_i)(x) = (f_1(x_1), f_2(x_2), \cdots, f_m(x_m))$$

for any element $x = (x_1, x_2, \cdots, x_m) \in \times F_i$. Then $\times f_i: \times F_i \to \times E_i$ is differentiable at the point $a = (a_1, a_2, \cdots, a_m) \in \times F_i$ if and only if for each i $(i = 1, 2, \cdots, m)$ $f_i: F_i \to E_i$ is differentiable at the point $a_i \in F_i$, and then

$$(\times f_i)'(a) = \times f_i'(a_i).$$

Proof. (a) Suppose that $\times f_i: \times F_i \to \times E_i$ is differentiable at the point $a = (a_1, a_2, \cdots, a_m) \in \times F_i$, i.e., there exists a map $\times l_i \in L(\times F_i; \times E_i)$ such that the map $\times r_i: \times F_i \to \times E_i$ defined by

$$(\times f_i)(a+h) = (\times f_i)(a) + (\times l_i)(h) + (\times r_i)(h)$$

is a remainder, where $h = (h_1, h_2, \cdots, h_m) \in \times F_i$.

$$\therefore \quad f_i(a_i + h_i) = f_i(a_i) + l_i(h_i) + r_i(h_i), \qquad i = 1, 2, \cdots, m.$$

We shall show that it follows from $\times l_i \in L(\times F_i; \times E_i)$ that for each i $(i = 1, 2, \cdots, m)$

$$l_i \in L(F_i; E_i).$$

In fact, by $\times l_i \in L(\times F_i; \times E_i)$,

$$(\times l_i)(h + h') = (\times l_i)(h) + (\times l_i)(h')$$

where $h = (h_1, h_2, \cdots, h_m)$, $h' = (h_1', h_2', \cdots, h_m')$ are arbitrary elements of $\times F_i$. From this we have

$$\begin{aligned}&(l_1(h_1 + h_1'), l_2(h_2 + h_2'), \cdots, l_m(h_m + h_m'))\\ &\quad = (l_1(h_1), l_2(h_2), \cdots, l_m(h_m)) + (l_1(h_1'), l_2(h_2'), \cdots, l_m(h_m'))\\ &\quad = (l_1(h_1) + l_1(h_1'), l_2(h_2) + l_2(h_2'), \cdots, l_m(h_m) + l_m(h_m')).\\ &\quad \therefore \quad l_i(h_i + h_i') = l_i(h_i) + l_i(h_i'), \qquad i = 1, 2, \cdots, m.\end{aligned}$$

That is, $l_1, l_2, \cdots, l_m$ are linear.

By $\times l_i \in L(\times F_i; \times E_i)$, $\times l_i$ is continuous, and therefore it is obvious that l_i is continuous.

$$\therefore \quad l_i \in L(F_i; E_i), \qquad i = 1, 2, \cdots, m.$$

We shall next show that $\times r_i \in R(\times F_i; \times E_i)$ implies $r_i \in R(F_i; E_i)$, $i = 1, 2, \cdots, m$.