95. *Calculus in Ranked Vector Spaces.* V

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(2.1.8) **Proposition.** *If E_2 is a separated ranked vector space, then the only remainder $r \in R(E_1; E_2)$ which is linear is the zero map.*

Proof. Let x be an arbitrary point of E_1 and consider a sequence $\{x_n\}$ such that $x_n = x$ for $n=0, 1, 2, \cdots$. Then by (1.7.3) $\{x_n\}$ is a quasi-bounded sequence. Let $\{\lambda_n\}$ be a sequence in $\mathfrak{R}$ with $\lambda_n \to 0$, then it follows from $r \in R(E_1; E_2)$ that

$$\left\{\lim \frac{r(\lambda_n x_n)}{\lambda_n}\right\} \ni 0.$$

The linearity of r implies

$$\frac{r(\lambda_n x_n)}{\lambda_n} = \frac{\lambda_n r(x_n)}{\lambda_n} = r(x_n)$$

$$\therefore \quad \{\lim r(x_n)\} \ni 0.$$

On the other hand, using $r(x_n) = r(x)$ for $n=0, 1, 2, \cdots$ and (1.2.4), we have

$$\{\lim r(x_n)\} \ni r(x).$$

Since E_2 is a separated ranked vector space, by (1.4.3)

$$r(x) = 0.$$

Hence $r: E_1 \to E_2$ is the zero map.

2.2. Differentiability at a point. In order to make use of (2.1.8) we assume henceforth that all spaces $E_1, E_2, \cdots$ are separated.

(2.2.1) **Proposition.** *Let $f: E_1 \to E_2$ be a map between ranked vector spaces E_1, E_2. If there exists a map $l \in L(E_1; E_2)$ such that the map $r: E_1 \to E_2$ defined by*

$$f(a+h) = f(a) + l(h) + r(h)$$

is a remainder, then l is uniquely determined.

Proof. Suppose that there exist two maps $l_1, l_2 \in L(E_1; E_2)$ such that the maps r_1, r_2 defined by

$$f(a+h) = f(a) + l_1(h) + r_1(h),$$
$$f(a+h) = f(a) + l_2(h) + r_2(h)$$

are remainders. Then we have

$$l_1(h) - l_2(h) = r_2(h) - r_1(h).$$

Since by (2.1.4) $R(E_1; E_2)$ is a vector space and by (2.1.5) $L(E_1; E_2)$ is also a vector space,

$$r_2 - r_1 \in R(E_1; E_2) \quad \text{and} \quad r_2 - r_1 \in L(E_1; E_2).$$