94. *On a Hardy's Theorem*

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1. Introduction and theorems. 1.1. Let f be an even and integrable function with period 2π and with mean value zero and let its Fourier series be

$$f(x) \sim \sum_{n=1}^{\infty} a_n \cos nx. \tag{1}$$

We suppose always $1<p<\infty$. By L^p we denote the space of such functions whose p-th powers are integrable. We put

$$A_n = \frac{1}{n} \sum_{k=1}^{n} a_k \qquad (n=1, 2, \cdots), \tag{2}$$

then Hardy [1] proved that there is an integrable function F such that

$$F(x) \sim \sum_{n=1}^{\infty} A_n \cos nx. \tag{3}$$

Further he [1] proved the following

Theorem I. $f \in L^p \Rightarrow F \in L^p$.

Petersen [2] has proved that the space L^p in Theorem I can be replaced by the Lorentz space Λ^p [3] which consists of even and integrable functions f with mean value zero such that

$$\int_0^{\pi} f^*(t) t^{-1/q} dt < \infty \qquad (1/p+1/q=1),$$

where f^* is the monotone decreasing rearrangement of $|f(t)|$. It is known that $\Lambda^p \subset L^p$ ([3], p. 39). Petersen's theorem[1)] is as follows:

Theorem II. $f \in \Lambda^p \Rightarrow F \in \Lambda^p$.

1.2. *Let L_0^p be the space of even and integrable functions f with mean value zero and with neighbourhood of the origin where the p-th power of $|f|$ is integrable.* Then Theorem I is generalized as follows:

Theorem I′. $f \in L_0^p \Rightarrow F \in L^p$.

We introduce another space M^p which consists of even and integrable functions f with mean value zero, satisfying the condition

$$\int_0^{\pi} |f(t)| t^{-1/q} dt < \infty \qquad (1/p+1/q=1),$$

(cf. [4]). Evidently $M^p \supset M^{p'}$ for $1<p<p'$. By Hölder's inequality we get

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1) Petersen has proved similar theorems for the other Lorentz spaces.