93. *Fourier Series of Functions of Bounded Variation*

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Let f be an integrable function with period 2π and let

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{1}$$

The following theorems are well known ([1], pp. 48, 57–58; [2] pp. 71–72, 114–116):

Theorem 1. *If f is of bounded variation, then*

$$|a_n| \leqq V/\pi n, \quad |b_n| \leqq V/\pi n \quad \textit{for all } n>1, \tag{2}$$

where V is the total variation of f over $(0, 2\pi)$.

Theorem 2. *If f is of bounded variation, then the Fourier series* (1) *converges to $\frac{1}{2}(f(x+0)+f(x-0))$ for every x.*

Recently, M. Taibleson [3] has given an elementary proof of Theorem 1, except the constant V/π in (2), which is the best possible. We shall give elementary proofs of Theorems 1 and 2.

Proof of Theorem 1.

$$\begin{aligned} \pi a_n &= \int_0^{2\pi} f(x)\cos nx\,dx = \int_{-\pi/2n}^{2\pi-\pi/2n} = \sum_{k=0}^{2n-1}\int_{(k-1/2)\pi/n}^{(k+1/2)\pi/n} \\ &= \sum_{k=0}^{2n-1}(-1)^k\int_{-\pi/2n}^{\pi/2n} f(x+k\pi/n)\cos nx\,dx \\ &= \int_{-\pi/2n}^{\pi/2n}\left[\sum_{j=0}^{n-1}(f(x+2j\pi/n)-f(x+(2j+1)\pi/n))\right]\cos nx\,dx \\ &= -\int_{-\pi/2n}^{\pi/2n}\left[\sum_{j=0}^{n-1}(f(x+(2j+1)\pi/n)-f(x+(2j+2)\pi/n))\right]\cos nx\,dx \end{aligned} \tag{3}$$

and then

$$\begin{aligned} 2\pi|a_n| &\leqq \int_{-\pi/2n}^{\pi/2n}\left[\sum_{k=0}^{2n-1}|f(x+k\pi/n)-f(x+(k+1)\pi/n)|\right]\cos nx\,dx \\ &\leqq V\int_{-\pi/2n}^{\pi/2n}\cos nx\,dx = 2V/n. \end{aligned}$$

Thus we get $|a_n| \leqq V/\pi n$. Similarly for b_n.

Proof of Theorem 2. We can suppose $f(x)=\frac{1}{2}[f(x+0)+f(x-0)]$ for all x. We put $f_x(t)=f(x+t)+f(x-t)-2f(x)$, then $f_x(t)$ is continuous at $t=0$. We denote by M the upper bound of $|f_x(t)|$ and by $V(a, b)$ the total variation of f_x on the interval (a, b), then we can easily see that