144. A Remark on a Problem of M. A. Naimark

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Gelfand and Naimark [6] characterized the algebra of all continuous complex-valued functions on a compact Hausdorff space as a commutative Banach *-algebra which satisfies the condition $||xx^*|| = ||x^*|| \cdot ||x||$: while Aren's generalization of the Gelfand-Naimark theorem is that a complete commutative seminormed *-algebra with a partition of unity is equivalent to the algebra of all continuous complex-valued functions on a locally compact paracompact space $C(T, K)$ [1]. A question is posed by Naimark in his treatise [6]: Is it possible to characterize all complete commutative seminormed *-algebras which are equivalent to (topologically equivalent to algebraically *-isomorphic) the algebras of all continuous complex-valued functions on locally compact Hausdorff spaces? Even though some more general result in this direction was obtained by Sha [6, 1964], it seems the problem remains open. Incidentally a solution of the problem was given by the writer [8, p. 182]. The purpose of this note is to present a modified proof of the solution and a second characterization in terms of seminorms.

"Seminormed algebra" and "locally multiplicatively convex algebra" (LMC) will be used synonymically in this paper. A subset Σ of an algebra is said to multiplicatively convex (m-convex) if $\Sigma\Sigma \subset \Sigma$. We assume the family $\mathcal{V}$ of seminorms of an algebra is so large that $V \in \mathcal{V}$, $U \leq V$ imply $U \in \mathcal{V}$. Some basic theorems and definitions employed hereafter are referred to [1], [3], and [4].

1. Functional representation. Lemma. If βX is the Stone-Čech compactification of a completely regular space X, then any unbounded continuous real function of f on X can be continuously extended to an extended function $\bar{f}$ over βX which admits $+\infty$ or $-\infty$ on some subsets of $\beta X - X$.

First proof. Let B be the two-point ($\pm\infty$) compactification of the real line. Then B is a compact Hausdorff space and f is a continuous mapping from X into B. By the Stone-Čech compactification theorem [2, p. 153] f has a continuous extension $\bar{f}$ on βX and the lemma is proved.

Second proof. Suppose