139. A Note on Inverse Images of Closed Mappings

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This paper is concerned with three results pertaining to the following problem. Given a mapping f in class C with the range of f in class $\mathcal{D}$, when will the domain of f be in class $\mathcal{E}$? In case f is a closed continuous mapping onto a paracompact Hausdorff space, S. Hanai [2, Theorem 5, p. 302] has given necessary and sufficient conditions for the domain of f to be normal. In Theorem 1, we provide another proof for Hanai's result, and in Theorem 2, under the same hypothesis on f, we obtain analagous necessary and sufficient conditions for the domain of f to be collectionwise normal. Under fairly restrictive hypothesis, Theorem 4 gives necessary and sufficient conditions for the domain of a mapping to be an M-space in the sense of Morita [6, p. 379].

In what follows, all mappings are assumed to be continuous. As usual, if X is a set, $\mathcal{F}=\{F_\alpha:\alpha\in A\}$ a collection of subsets of X, and $S\subseteq X$, we let $\mathcal{F}|S=\{F_\alpha\cap S:\alpha\in A\}$.

Let f be a mapping from X to the T_1 space Y, C a closed subset of X, and m a cardinal number. f satisfies condition γ_m at C iff for any discrete collection $\{C_\alpha:\alpha\in A\}$ of $\leq m$ closed subsets of C, there exists a pairwise disjoint open collection $\{U_\alpha:\alpha\in A\}$ such that $C_\alpha\subseteq U_\alpha$ for all α. If f satisfies condition γ_m at C for all cardinals m, we say that f satisfies condition γ at C.

Lemma 1.1. *Let f be a closed mapping from the topological space X onto the T_1 regular space Y. Suppose that f satisfies condition γ_2 at $f^{-1}(y)$ for all y in Y. Then for any y in Y, closed subset C of $f^{-1}(y)$, and open set U containing C, there exists an opeu set V such that $C\subseteq V\subseteq\overline{V}\subseteq U$.*

Proof. Let the closed set C of $f^{-1}(y)$ be contained in the open set 0. Using condition γ_2, choose open sets W_1 and W_2 of X containing C and $(X-0)\cap f^{-1}(y)$ respectively. Then $K=(X-0)-W_2$ is closed and misses $f^{-1}(y)$. Hence by regularity of Y, choose an open set P of Y with $y\in P\subseteq\overline{P}\subseteq Y-f(K)$. If $V=W_1\cap f^{-1}(P)$, then V is as desired.

Theorem 1. *Let f be a closed mapping from the topological space X onto the paracompact Hausdorff space Y. X is normal iff f satisfies condition γ_2 at $f^{-1}(y)$ for all y in Y.*