135. Notes on the Uniform Distribution of Sequences of Integers

By Lauwerens KUIPERS*) and Saburô UCHIYAMA**)

(Comm. by Zyoiti SUETUNA, M. J. A., Sept. 12, 1968)

In 1961 I. Niven [5] introduced the following concept of uniform distribution of sequences of integers. Let $A=(a_n)$ be an infinite sequence of integers not necessarily distinct from each other. For any integers j and $m\geqq 2$ we denote by $A(N, j, m)$ the number of terms a_n $(1\leqq n\leqq N)$ satisfying the condition $a_n\equiv j \pmod m$. The sequence A is said to be uniformly distributed (mod m) if the limit

$$\lim_{N\to\infty}\frac{1}{N}A(N, j, m)=\frac{1}{m}$$

exists for all j, $1\leqq j\leqq m$. If A is uniformly distributed (mod m) for every integer $m\geqq 2$, A is said to be uniformly distributed.

S. Uchiyama [9] has proved the following theorem which is the analogue of the Weyl criterion:

Theorem 1. *Let $A=(a_n)$ be an infinite sequence of integers. A necessary and sufficient condition that A be uniformly distributed* (mod m), $m\geqq 2$, *is that*

$$\lim_{N\to\infty}\frac{1}{N}S_N\left(A,\frac{h}{m}\right)=0$$

for all h, $1\leqq h\leqq m-1$, where

$$S_N(A, t)=\sum_{n=1}^{N}e(a_nt), \qquad e(t)=e^{2\pi it}.$$

Hence:

Corollary 1. *A necessary and sufficient condition for an infinite sequence $A=(a_n)$ of integers to be uniformly distributed is that*

$$\lim_{N\to\infty}\frac{1}{N}S_N(A, t)=0$$

for all rational numbers t, $t\not\equiv 0$ (mod 1).

In order to prove Theorem 1 it will suffice to observe that

$$\sum_{h=1}^{m-1}\left|S_N\left(A, \frac{h}{m}\right)\right|^2=m\sum_{j=1}^{m}\left(A(N, j, m)-\frac{N}{m}\right)^2.$$

The notion of uniform distribution of integers has been generalized by H. G. Meijer [4] to the notion of uniform distribution of g-adic

---

*) Department of Mathematics, Southern Illinois University, Carbondale, Ill., U. S. A.

**) Department of Mathematics, Shinshû University, Matsumoto, Japan.