134. On Some Trigonometric Series

By Masako IZUMI and Shin-ichi IZUMI
Department of Mathematics, The Australian National University, Canberra, Australia

(Comm. by Zyoiti SUETUNA, M. J. A., Sept. 12, 1968)

1. Introduction and Theorems. 1.1. Let us consider a trigonometric series

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx). \tag{1}$$

If we write $r_n=\sqrt{a_n^2+b_n^2}$, then the series (1) can be written in the form

$$\sum_{n=0}^{\infty} r_n\cos(nx+c_n),\quad -\pi/2\leqq c_n<3\pi/2. \tag{2}$$

We denote by α_0 the root of the equation

$$\int_0^{3\pi/2} x^{-\alpha}\cos x\,dx=0.$$

It is known that $\alpha_0=0.30844\cdots$.

In the case $c_n=0$ $(n=1,2,\cdots)$ in (2), we have proved the following theorems [1], as generalization of Chowla's and Selberg's theorems [2].

Theorem I. *If the series* $\sum_{n=1}^{\infty} r_n/n^{1-\beta}$ *diverges for a* β, $0\leqq\beta<\alpha_0$, *then*

$$\limsup_{N\to\infty}\left\{-\min_{0\leqq x<\pi}\left(\sum_{n=1}^{N} r_n\cos nx\Big/\sum_{n=1}^{N} r_n\right)\right\}\geqq A(\beta)$$

where $A(\beta)=-(1-\beta)(2/(3\pi))^{1-\beta}\int_0^{3\pi/2}x^{-\beta}\cos x\,dx>0$. *The constant* $A(\beta)$ *is the best possible one.*

Theorem II. *Let* $0\leqq\beta<\alpha_0$. *If there exist* $A>0$ *and* λ, $0\leqq\lambda<1-\beta$ *such that*

$$\sum_{n=1}^{N} r_n\cos nx\geqq -AN^{\lambda}\quad \textit{for all } x \textit{ and all } N,$$

then the series $\sum_{n=1}^{\infty} r_n/n^{1-\beta}$ *converges.*

1.2. For non-vanishing (c_n), Chidambaraswamy and Shah [3] have proved the following generalization of Chowla's and Selberg's theorems [2].

Theorem III. *If* $r_0>0$,

$$\sum_{n=0}^{N} r_n\cos(nx+c_n)\geqq 0\quad \textit{for all } x \textit{ and all } N$$