129. Rings of Dominant Dimension $\geqq 1$

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Tachikawa [9] was the first to introduce and to study dominant dimensions for algebras. In his paper [9], dominant dimensions play a vital role in classifying QF-3 algebras. The author has found it relevant, in the study of rings with torsionless injective hulls, to introduce new dominant dimension defined by the analogue of Tachikawa [9, p. 249] using torsionless minimal injective resolutions.

In Section 1, we introduce dominant dimension for torsionless modules and then give illustrative examples. In Section 2, we are concerned with rings having dominant dimension $\geqq 1$ and show how to obtain such rings. In Theorem 1, (3), we construct a ring Q having dominant dimension ∞ such that Q_Q is a non-injective non-cogenerator in the category of right Q-modules $\mathcal{M}_Q$. The endomorphism rings of generator-cogenerators are discussed in Section 3. Let R and Q be rings. Denote by $\mathcal{M}_R$ (resp. $\mathcal{L}_Q$) the category of right R-modules (resp. the category of right Q-modules having dominant dimension $\geqq 2$). Then our main Theorem 2 states that Q is the endomorphism ring of a generator-cogenerator in $\mathcal{M}_R$ if and only if Q_Q has dominant dimension $\geqq 2$ and there exists an equivalence $\mathcal{M}_R \sim \mathcal{L}_Q$.

In this paper, rings will have a unit element and modules will be unital. If A_R is a module over a ring R, $E(A_R)$ will denote the injective hull of A_R, and $\mathrm{End}(A_R)$ the endomorphism ring of A_R. We adopt the following notational trick, which will facilitate our study: homomorphisms will be written on the side opposite the scalars.

1. Dominant dimension. Let R be a ring, A_R a right R-module, and let

$$0 \to A \to X_1 \to X_2 \to \cdots \to X_n$$

be a minimal (essential) injective resolution of A. We shall say that A has dominant dimension $\geqq n$ if each X_i is torsionless (denoted by domi. dim. $A \geqq n$). In case domi. dim. $A \geqq n$ and domi. dim. $A \ngeqq n+1$, domi. dim. $A = n$. If domi. dim. $A \geqq n$ for each n, domi. dim. $A = \infty$. In case domi. dim. $A \ngeqq 1$, domi. dim. $A = 0$. In case R is left Artinian, torsionless injective right R-modules are always projective by Chase [3, Theorem 3.3] and hence our dominant dimension exactly coincides with that of Tachikawa [9, p. 249]. We consider some illustrative examples of dominant dimensions: