128. A Milnor Conjecture on Spin Structures

By Seiya SASAO
Department of Mathematics, Tokyo Institute of Technology

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Let ξ denote a principal $SO(n)$-bundle over a CW-complex B and let $E(\xi)$ denote the total space of ξ. A spin structure on ξ is a pair (η, f) which satisfies

(1) A principal bundle η over B with the spinor group Spin(n) as structural group; and

(2) A map $f: E(\eta)\to E(\xi)$ such that the following diagram is commutative.

$$\begin{array}{ccc} E(\eta)\times \mathrm{Spin}(n) & \rightarrow & E(\eta) \\ \downarrow {\scriptstyle f\times\lambda} & & \downarrow {\scriptstyle f} \\ E(\xi)\times SO(n) & \longrightarrow & E(\xi) \end{array} \begin{array}{c} \searrow \\ \nearrow \end{array} B.$$

Here λ denotes the standard homomorphism from Spin(n) to $SO(n)$ and horizontal lines denote the right translation. A second spin structure (η', f') on ξ is identified with (η, f) if there exists an isomorphism g from η' to η so that $f\circ g=f'$. Then J. Milnor stated the following conjecture [1, pp. 198–203]:

If (η, f) and (η', f') are two spin structures on the same $SO(n)$-bundle, with $n>\dim B$, then η is necessarily isomorphic to η'.

In this note we shall present the affirmative answer when B is compact connected. By Milnor we have the following

Lemma [1, p. 199]: If ξ admits a spin structure then the number of distinct spin structures on ξ is equal to the number of elements in $H^1(B; Z_2)$.

Now the following lemma is clear.

Lemma 1. *If ξ admits two spin structures (η, f) and (η', f') such that η is isomorphic to η' then there exists a spin structure (η, f'') on ξ which is isomorphic to (η', f').*

Let p_ξ denote the projection map of the bundle ξ. If two spin structures (η, f_1), (η, f_2) are given, from $p_\eta=p_\xi f_1=p_\xi f_2$, we have a map $g: E(\eta)\to SO(n)$ defined by $f_1(x)=f_2(x)\cdot g(x)$ for $x\in E(\eta)$. Here $\cdot$ denotes the right translation. Clearly g satisfies $g(x\cdot h)=\lambda(h)^{-1}\times g(x)\times\lambda(h)$ for $h\in \mathrm{Spin}(n)$ where $\times$ denotes the group multiplication. Conversely g is a map as above and let (η, f) be a spin structure on ξ. Then $(\eta, f\cdot g)$[1] is also a spin structure on ξ. And moreover let g' be another map such as g. Then $(\eta, f\cdot g)$ is isomorphic to $(\eta, f\cdot g')$ if

1) Of course the map $f\cdot g$ is defined by $(f\cdot g)(x)=f(x)\cdot g(z)$.