179. *A Metric Characterization of the Cartesean Decomposition in a ☆-Algebra*

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1. Ky Fan and A. J. Hoffman [2] observed, among others:

If T is an $n\times n$ matrix and if an $n\times n$ matrix A satisfies $A=\mathrm{Re}\,T$, then

(1) $$\|T-A\|_*\leqq\|T-H\|_*$$

for any hermitean $n\times n$ matrix H, where $\|C\|_*$ is a unitarily invariant norm of C.

Very recently, the theorem of Fan and Hoffman is generalized for an operator T belonging to a finite factor by Marie and Hisashi Choda [1] under the restriction that the norm is defined by

(2) $$\|C\|_*^2=\varphi(C^*C),$$

where φ is the trace of factor. But the norm the condition (2) is too restrictive so that the theorem of Fan and Hoffman is excluded.

In the present note, we shall give an abstract formulation which includes the both of the theorems of Fan-Hoffman and Choda. Through this formulation, we shall show that the self adjoint operator A in the Cartesean Decomposition is the nearest self adjoint opertor to the given T in ☆-algebra $\mathfrak{A}$, which will give a metric characterization of the Cartesean Decomposition in ☆-algebra $\mathfrak{A}$.

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2. Throughout this note, we shall assume that $\mathfrak{A}$ be a linear space with an involution $x\to x^*$([3])

I 1 $$(\alpha x+\beta y)^*=\alpha^*x^*+\beta^*y^*,$$

I 2 $$x^{**}=x,$$

where α^* is the complex conjugate of α. An element T of $\mathfrak{A}$ will be called self adjoint or hermitean if $T^*=T$. It is easy to deduce that the set $\mathfrak{A}^s$ of all hermitean members of $\mathfrak{A}$ is a real linear subspace of $\mathfrak{A}$, whence $\mathfrak{A}^s$ is convex. Let T be an element of $\mathfrak{A}$, then we have the cartesean decomposition of T by

(3) $$T=\mathrm{Re}\,T+i\,\mathrm{Im}\,T,$$

where Re T and Im T are self adjoint which are defined by

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