171. Notes on Medial Archimedean Semigroups without Idempotent

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1. **Introduction.** A semigroup S is called medial if S satisfies the identity $xyzu=xzyu$. According to Chrislock [1], [2], a medial semigroup S is $\mathcal{S}$-indecomposable (or $\mathfrak{p}$-simple), that is, having no semilattice-homomorphic image except a trivial one, if and only if S satisfies: for every $a, b \in S$ there are $x, y, z, u \in S$ and positive integers m and n such that

$$a^m = xby \quad \text{and} \quad b^n = zau.$$

This property is called archimedeaness which coincides with "archimedeaness" [3] in commutative semigroups.

The author proved the following theorem (cf. [4]):

Theorem 1. *If S is a commutative archimedean semigroup without idempotent, the closet*[1] *is empty for all elements, that is,*

$$(1) \qquad \bigcap_{n=1}^{\infty} a^n S = \phi \quad \textit{for all } a \in S.$$

In this note we will extend this theorem to medial semigroups and will state its various applications.

Theorem 2. *If S is a medial archimedean semigroup without idempotent, then*

$$(2) \qquad \bigcap_{n=1}^{\infty} Sa^n S = \phi \quad \textit{for all } a \in S.$$

Proof. Let $D=\bigcap_{n=1}^{\infty} Sa^nS$ and suppose that $D \neq \phi$. Then $aDa \neq \phi$ for all $a \in S$. By mediality

$$(3) \qquad aDa \subseteq \bigcap_{n=1}^{\infty} aSa^nSa = \bigcap_{n=1}^{\infty} a^n aS^2 a \subseteq \bigcap_{n=1}^{\infty} a^n aSa = \bigcap_{n=1}^{\infty} a^{3m} aSa.$$

On the other hand, aSa is obviously a subsemigroup and it is commutative since

$$(axa)(aya) = (aya)(axa) \quad \text{for all } x, y \in S.$$

We will prove that aSa is archimedean. Since S is medial archimedean, for axa and aya, there are $u, v \in S$, and a positive integer k such that

$$(axa)^k = u(aya)v.$$

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1) $\bigcap_{n=1}^{\infty} a^n S$ is called the closet of a. See [5].