170. On Extensions with Given Ramification

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Let k be a number field of finite degree, and let S be a set of primes of k, including the achimedean ones. Let G be the Galois group of the maximal extension Ω of k unramified outside S. Throughout this paper we assume that S contains all primes above a fixed prime number l. Tate [7] has asserted that G has strict cohomological dimension 2 with respect to l, if k is totally imaginary in case $l=2$, but the proof has been unpublished. (Brumer [3] showed that G has cohomological dimension 2 with respect to l under the same assumptions.) We shall give here the proof of the above Tate's theorem (Section 1). As a corollary of this theorem, we obtain an arithmetic theorem and we get the l-adic independence of independent units (Section 2). Finally, we shall determine the structure of the connected component of the S-idèle class group. This is a generalization of the results of Weil [10] and Artin [1] (see also [2; Chap. IX]).

1. Cohomological dimension. Throughout this paper notations and terminologies are the same as in Tate [7]. By m we shall always understand a positive integer such that $mk_S=k_S$ where k_S is the ring of all S-integers of k. For any abelian group A, let $A(l)$ denote the l-torsion part of A. Let μ denote the group of all roots of unity, and let μ_m denote the group of m-th roots of unity.

Theorem 1. *Let $\bar{J}^S$ denote the projection to S_0 of the idèle group of Ω, where S_0 is the set of non-archimedean primes in S. We put $E=\bar{J}^S(l)/\mu(l)$. Suppose that k is totally imaginary if $l=2$. Then, for any l-torsion module M, we have an isomorphism*

$$H^2(k_S, M)^*\cong \mathrm{Hom}_G(M, E).$$

Proof. By our assumptions G has cohomological l-dimension 2. Let $\bar{E}$ be a *module dualisant* for G with respect to l. We shall show $E=\bar{E}$. By [5; Chap. I, Annexe] we have $\bar{E}=\varinjlim_{t} D_2(\boldsymbol{Z}/l^t\boldsymbol{Z})$ where $D_2(\boldsymbol{Z}/m\boldsymbol{Z})=\varinjlim_{K\subset\Omega} H^2(K_S, \boldsymbol{Z}/m\boldsymbol{Z})^*$, the inductive limit is taken with respect to cores*. By Tate's duality theorem, we have a commutative exact diagram