169. *On Lacunary Trigonometric Series. II*

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§1. Introduction. In [3] we have proved

Theorem A. *Let $\{n_k\}$ be a sequence of positive integers and $\{a_k\}$ a sequence of non-negative real numbers for which the conditions*

(1.1) $$n_{k+1}>n_k(1+ck^{-\alpha}), \qquad k=1, 2, \cdots,$$

(1.2) $$A_N=(2^{-1}\sum_{k=1}^{N} a_k^2)^{1/2}\to+\infty, \qquad \textit{as } N\to+\infty,$$

and

(1.3) $$a_N=o(A_N N^{-\alpha}), \qquad \textit{as } N\to+\infty,$$

are satisfied, where c and α are any given constants such that

(1.4) $$c>0 \qquad \textit{and} \qquad 0\le\alpha\le 1/2.$$

Then we have, for all x,

(1.5) $$\lim_{N\to\infty}|\{t\,;\,t\in E,\ \sum_{k=1}^{N} a_k\cos 2\pi n_k(t+\alpha_k)\le xA_N\}|\,/\,|E| = (2\pi)^{-1/2}\int_{-\infty}^{x}\exp(-u^2/2)du,^{*)}$$

where $E\subset[0, 1]$ is any given set of positive measure and $\{\alpha_k\}$ any given sequence of real numbers.

This theorem was first proved by R. Salem and A. Zygmund in case of $\alpha=0$, where $\{n_k\}$ satisfies the so-called *Hadamard's gap* condition (cf. [4], (5.5), pp. 264–268). In that case they also remarked that under the hypothesis (1.2) the condition (1.3) is *necessary* for the validity of (1.5) (cf. [4], (5.27), pp. 268–269).

Further, in [2] P. Erdös has pointed out that for every positive constant c there exists a sequence of positive integers $\{n_k\}$ such that $n_{k+1}>n_k(1+ck^{-1/2})$, $k\ge 1$, and (1.5) is not true for $a_k=1$, $k\ge 1$, and $E=[0, 1]$. But I could not follow his argument on the example.

The purpose of the present note is to prove the following

Theorem B. *For any given constants $c>0$ and $0\le\alpha\le 1/2$, there exist sequences of positive integers $\{n_k\}$ and non-negative real numbers $\{a_k\}$ for which the conditions* (1.1), (1.2) *and*

(1.6) $$a_N=O(A_N N^{-\alpha}), \qquad \textit{as } N\to+\infty,$$

are satisfied, but (1.5) *is not true for $E=[0, 1]$ and $\alpha_k=0$, $k\ge 1$.*

The above theorem shows that in Theorem A the condition (1.3) is

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*) $|E|$ denotes the Lebesgue measure of a set E.