212. A Note on Traces on von Neumann Algebras

By Yoshinori HAGA
Fuculty of Engineering, Ibaraki University

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The purpose of this note is to show a theorem concerning traces on von Neumann algebras, motivated by a theorem of Kakutani [4] on divergent integrals. Our theorem may be seen as an extension of Kakutani's theorem to the non-commutative abstract integral theory.

Let M^+ be the set of all positive elements of a von Neumann algebra M. A *trace* on M^+ is a functional φ defined on M^+, with values $\geqq 0$, finite or infinite, having the following properties:

(i) If $S, T \in M^+$, $\varphi(S+T)=\varphi(S)+\varphi(T)$.

(ii) If $S \in M^+$ and λ is a number $\geqq 0$, $\varphi(\lambda S)=\lambda\varphi(S)$ (here we define $0\cdot(+\infty)=0$).

(iii) If $S \in M^+$ and U is unitary, $\varphi(USU^{-1})=\varphi(S)$.

We say φ is *finite* if $\varphi(S)<+\infty$ for all $S \in M^+$, and φ is *normal* if $\varphi(\sup S_i)=\sup \varphi(S_i)$ for every uniformly bounded increasing directed set (S_i) in M^+.

Theorem. *Let M be a von Neumann algebra, and φ and ψ be normal traces on M^+. Suppose that*

(1) $$\psi(S)<+\infty \quad \textit{implies} \quad \varphi(S)<+\infty.$$

Then, there exist a positive constant K and a finite normal trace τ on M^+ such that

(2) $$\varphi(S)\leqq K\psi(S)+\tau(S) \quad \textit{for any} \quad S \in M^+.$$

This theorem concerns essentially with semi-finite von Neumann algebras because we assume the existence of normal traces, but we state and prove it without any restrictions of the types of M.

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1. Preliminary results. M^P and M^U denote the sets of all projections and unitary operators of a von Neumann algebra M respectively. Let $E, F \in M^P$. If there is a partially isometric $V \in M$ such that $V^*V=E$ and $VV^*=F$, we say E and F are *equivalent* and denote by $E\sim F$. If there is a projection F_1 such that $E\sim F_1\leqq F$, we write $E\prec F$. Let $(E_i)_{i\in I}$ (resp. $(F_i)_{i\in I}$) be a family of mutually orthogonal projections in M, and let $E=\sum_{i\in I} E_i$ (resp. $F=\sum_{i\in I} F_i$), then E and F are also projections in M. Moreover, if $E_i\sim F_i$ (resp. $E_i\prec F_i$) for all $i\in I$,